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Weighted Hardy-type inequalities

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Declaration

I confirm that this thesis is the result of my own research and that all borrowed ideas and materials have been properly cited

Atina Alga

June 2026

Acknowledgements

I would like to express my deepest gratitude to my scientific advisor for answering all of my silly questions, explaining everything patiently without getting angry, and giving me valuable advice. I would also like to express my sincere gratitude to my university for giving me this opportunity, as well as to my teachers and parents for their support.

Dedication

This thesis is dedicated to my future, to my dreams, and to the path of knowledge and personal growth that I continue to follow.

Abstract

This thesis is devoted to weighted Hardy-type inequalities and identities obtained by the factorization method. The main focus is on Hardy-type identities in the Baouendi–Grushin setting, where the Euclidean radial derivative is replaced by the radial derivative associated with the Grushin structure.

More precisely, we consider the homogeneous distance function ρ and the Grushin gradient ∇_γ, and study identities involving the radial Grushin derivative

$$\frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|}.$$

Using suitable first-order differential operators and their formal adjoints, we derive an improved weighted Hardy identity with an explicit non-negative remainder term. The result is formulated under the assumption that the weight functions V and W form a Bessel pair adapted to the homogeneous dimension Q.

The obtained identity extends classical radial Hardy-type identities with Bessel pairs to the degenerate Baouendi–Grushin framework. Several special cases are also discussed, including power-type and logarithmic weights, which illustrate how the general theorem leads to concrete weighted Hardy-type inequalities.

Аңдатпа

Бұл диссертациялық жұмыс факторизация әдісі арқылы алынатын салмақты Hardy типтес теңсіздіктер мен тепе-теңдіктерге арналған. Негізгі назар Baouendi--Grushin жағдайындағы Hardy типтес тепе-теңдіктерге аударылады, мұнда евклидтік радиал туынды Grushin құрылымымен байланысты радиал туындымен алмастырылады.

Нақтырақ айтқанда, біз біртекті қашықтық функциясы ρ мен Grushin градиенті ∇_γ қарастырамыз және радиал Grushin туындысын қамтитын тепе-теңдіктерді зерттейміз

$$\frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|}.$$

Сәйкес бірінші ретті дифференциалдық операторлар мен олардың формалды түйіндестерін қолдану арқылы біз теріс емес қалдық мүшесі бар жақсартылған салмақты Hardy тепе-теңдігін аламыз. Нәтиже V және W салмақ функциялары біртекті өлшем Q-ге бейімделген Bessel pair құрайды деген шартпен тұжырымдалады.

Алынған тепе-теңдік Bessel pair арқылы берілген классикалық радиал Hardy типтес тепе-теңдіктерді Baouendi--Grushin құрылымына жалпылайды. Сонымен қатар бірнеше дербес жағдайлар қарастырылады, соның ішінде дәрежелік және логарифмдік салмақтар, олар жалпы теореманың нақты салмақты Hardy типтес теңсіздіктерге қалай алып келетінін көрсетеді.

Аннотация

Данная диссертационная работа посвящена весовым неравенствам и тождествам типа Харди, полученным методом факторизации. Основное внимание уделяется тождествам типа Харди в постановке Baouendi--Grushin, где евклидова радиальная производная заменяется радиальной производной, связанной со структурой Grushin.

Более точно, мы рассматриваем однородную функцию расстояния ρ и градиент Grushin ∇_γ, а также изучаем тождества, содержащие радиальную производную Grushin

$$\frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|}.$$

Используя подходящие дифференциальные операторы первого порядка и их формально сопряжённые операторы, мы выводим улучшенное весовое тождество Харди с явным неотрицательным остаточным членом. Результат формулируется при условии, что весовые функции V и W образуют Bessel pair, адаптированную к однородной размерности Q.

Полученное тождество обобщает классические радиальные тождества типа Харди с Bessel pair на вырожденную структуру Baouendi--Grushin. Также рассматриваются несколько частных случаев, включая степенные и логарифмические веса, которые показывают, как общая теорема приводит к конкретным весовым неравенствам типа Харди.

Abbreviations

PDE – Partial Differential Equation

ODE – Ordinary Differential Equation

LHS – Left-hand side

RHS – Right-hand side

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Chapter 1

Background and motivations

1.1 Introduction

The Hardy-type inequalities have played a significant role in mathematical analysis, partial differential equations, spectral theory, and the theory of inequalities. After the original Hardy's inequality, there were many generalizations obtained that became a powerful means of investigating the properties of differential operators. Among other things, Hardy-type inequalities have been extensively applied in order to study the properties of functions at singularities, coercivity of differential operators, stability, and spectral properties of elliptic and subelliptic equations.

The classical Hardy inequality provides us with a precise inequality that relates the integral of the gradient of a function with the integral of the product of the function and a singular potential. Unfortunately, in many cases of applications of mathematical analysis, physics, and geometry, the classical Hardy inequality proves to be inadequate for our purposes. Thus, we are naturally led to considering generalized weighted Hardy-type inequalities.

In particular, inequalities of this kind are particularly relevant when there is a non-Euclidean geometry on the space or degeneracy present in the differential operator. This happens, for instance, in the case of homogeneous groups, stratified groups, and subelliptic problems. In such cases, the ordinary Euclidean norm and gradient have to be replaced with some homogeneous quasi-norms and subelliptic gradients. Thus, it becomes necessary to use a more general form of the Hardy inequality.

One of the key areas of research for improving the Hardy-type inequalities includes obtaining enhanced versions of inequalities with sharp remainder estimates. In addition to providing lower estimates, they also yield precise identities in which the difference between the energy part and the Hardy potential part is estimated by a positive remainder term. The importance of such identities lies in the fact that they establish the sharpness of constants and indicate the inner structure of the inequality.

The present dissertation studies weighted Hardy-type inequalities by using

the factorization method. In the latter, one considers appropriate first order differential operators along with their formal adjoints in order to arrive at Hardy-type identities in a systematic way. The construction of Bessel pairs allows for the identification of the weight functions in the potential part of the Hardy inequality and leads to improved Hardy-type identities with explicit remainder terms.

The key focus of this research is to explore the class of weighted Hardy-type inequalities along with their refinement versions. The emphasis is laid on identities containing weighted radial derivatives and the significance of the related Bessel-pair equation. These findings extend some existing Hardy-type inequalities and give rise to obtaining precise weighted inequalities with non-negative error terms.

1.1.1 Historical Development of Hardy-Type Inequalities

The history described in this subsection is based on the introduction to the book by Ruzhansky and Suragan [37]. The topic of Hardy inequalities is significant within modern mathematical analysis, and has remained a popular subject for study for more than a hundred years since Hardy formulated one of the basic inequalities of this class back in 1918. Over this time, this topic has turned into an extensive discipline, enjoying numerous applications in different areas of analysis and other fields of mathematics, including spectral theory and partial differential equations.

In its turn, origins of Hardy inequalities can be traced back to results of Hilbert on integral equations. The object of study of Hilbert was an interesting feature of convergence of series of the form

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n}, \tag{1.1.1}$$

where $a_n \geq 0$. This result later attracted the attention of several mathematicians, including H. Weyl, F. Wiener, and I. Schur, who gave different proofs of Hilbert's theorem. However, G. H. Hardy sought a simpler and more direct approach. In 1918, he obtained a new proof and showed that Hilbert's theorem could be derived as a consequence of the following statement: if

$$\sum_{m=1}^{\infty} a_m^2 \tag{1.1.2}$$

is convergent and

$$A_n = a_1 + \cdots + a_n, \tag{1.1.3}$$

then the series

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^2 \tag{1.1.4}$$

is also convergent. This result is often regarded as the starting point of what is

now known as Hardy's inequalities.

Having conveyed his proof to Marcel Riesz, the latter then suggested yet another generalization to the Hardy inequality. That is, if $p > 1$ and

$$\sum_{m=1}^{\infty} a_m^p \tag{1.1.5}$$

is convergent, then

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p \tag{1.1.6}$$

is also convergent. This result became one of the foundations of L^p-Hardy inequalities. In this sense, the development of Hardy inequalities is also closely related to the Hardy–Riesz type inequalities.

In addition to establishing the integral formulation of the inequality and finding the exact value of the optimal constant, Hardy also stated one of the most well-known versions of the inequality as

$$\int_a^{\infty} \left(\frac{1}{x} \int_a^x f(t)\, dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_a^{\infty} f^p(x)\, dx, \tag{1.1.7}$$

where $p > 1$, and the function f is positive. This integral inequality became a fundamental tool in the further development of functional inequalities.

Since the last hundred years, various forms of Hardy's inequality have been obtained. Some examples of these inequalities include Hardy inequalities with weights, higher order Hardy inequalities, L^p inequalities, Sobolev inequalities, Rellich inequalities, Caffarelli–Kohn–Nirenberg inequalities, and many others. Hardy inequalities have significant applications in the theory of differential operators, spectral inequalities, uncertainty inequalities, and partial differential equations.

In order to appreciate the analytical significance of Hardy-type inequalities, we must first revisit their classical versions and the results stemming from them. In modern mathematics, the L^2 classical Hardy inequality in the Euclidean setting $\mathbb{R}^n$ for $n \geq 3$ takes the form

$$\left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{n-2} \|\nabla f\|_{L^2(\mathbb{R}^n)}. \tag{1.1.8}$$

Here ∇ denotes the standard gradient in $\mathbb{R}^n$, and

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}$$

is the Euclidean norm. The function f is usually assumed to belong to $C_0^{\infty}(\mathbb{R}^n)$, and the constant

$$\frac{2}{n-2}$$

is known to be sharp. This means that the constant cannot be replaced by a smaller one while preserving the validity of the inequality for all admissible functions.

Hardy's inequality, being one of the most prominent in mathematics, has many important applications in various fields. The classical Hardy inequality has an application in spectral theory; specifically, it is useful in providing lower bounds for quadratic forms that come from the Laplacian operator. Also, Hardy inequalities have strong relations to uncertainty principles, Sobolev inequalities, and PDEs.

One of the important consequences related to Hardy's inequality is the uncertainty principle. The simplest way to represent it in the Euclidean space would be

$$\left(\int_{\mathbb{R}^n} |\nabla\phi|^2 dx\right)\left(\int_{\mathbb{R}^n} |x|^2\phi^2 dx\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{R}^n} \phi^2 dx\right)^2, \tag{1.1.9}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$ is a real-valued function. The constant $\frac{n^2}{4}$ is also sharp. This shows that Hardy inequalities are not only independent functional inequalities, but also serve as a powerful tool for deriving other fundamental estimates in analysis.

An analogous result is true for L^p-spaces as well. In particular, for $n \geq 2$ and $1 \leq p < n$, we have the L^p-Hardy inequality

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\|\nabla f\|_{L^p(\mathbb{R}^n)}. \tag{1.1.10}$$

Here $f \in C_0^\infty(\mathbb{R}^n)$, and the constant

$$\frac{p}{n-p}$$

is sharp. This generalization demonstrates that Hardy inequalities naturally arise not only in Hilbert spaces, but also in a wider scale of Lebesgue spaces.

One of the other interesting extensions of Hardy's inequality is called Rellich inequality, which in Euclidean space $\mathbb{R}^n$, where $n \geq 5$, is represented as

$$\left\|\frac{f}{|x|^2}\right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)}\|\Delta f\|_{L^2(\mathbb{R}^n)}. \tag{1.1.11}$$

In this sense, one could call this inequality an analogue of the Hardy inequality because it contains the Laplace operator Δf rather than ∇f. Rellich type inequalities are important in the study of higher order differential equations and their corresponding boundary value problems.

Hardy inequalities of higher order have also been well explored. For functions

$f \in C_0^k(0,\infty)$, one can consider inequalities of the form

$$\left\| \frac{f}{x^k} \right\|_{L^2(0,\infty)} \leq \frac{2^k}{(2k-1)!!} \left\| f^{(k)} \right\|_{L^2(0,\infty)}, \qquad k \in \mathbb{N}, \tag{1.1.12}$$

where

$$(2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1.$$

For $k = 1$, this inequality reduces to the one-dimensional Hardy inequality, while for $k = 2$, it is related to the one-dimensional Rellich inequality.

Thus, Hardy inequalities together with their various extensions constitute an indispensable component of modern analysis. With their optimal constants, weights, higher order versions, and relationships to differential operators, Hardy inequalities provide an important means for the study of functional inequalities and partial differential equations. The study of weighted Hardy-type inequalities is thus our motivation in this dissertation.

1.1.2 Fundamental Concepts and Definitions

Let (X, μ) be a measure space. The classical Lebesgue space $L^p(X)$, $1 \leq p < \infty$, is defined by

$$L^p(X) = \left\{ f : \int_X |f(x)|^p \, d\mu(x) < \infty \right\}. \tag{1.1.13}$$

The norm in $L^p(X)$ is given by

$$\|f\|_{L^p(X)} = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p}. \tag{1.1.14}$$

Weighted Lebesgue spaces are defined by introducing a measurable weight function $w(x) > 0$. The weighted space $L^p_w(X)$ consists of all measurable functions satisfying

$$\int_X |f(x)|^p w(x) \, d\mu(x) < \infty. \tag{1.1.15}$$

The corresponding norm is

$$\|f\|_{L^p_w(X)} = \left(\int_X |f(x)|^p w(x) \, d\mu(x) \right)^{1/p}. \tag{1.1.16}$$

One of the main operators studied in this dissertation is the Hardy operator

$$(Hf)(x) = \frac{1}{x} \int_0^x f(t) \, dt, \qquad x > 0. \tag{1.1.17}$$

The dual Hardy operator is defined by

$$(H^*f)(x) = \int_x^\infty \frac{f(t)}{t}\,dt. \tag{1.1.18}$$

The classical Hardy inequality states that

$$\|Hf\|_{L^p(0,\infty)} \le \frac{p}{p-1}\|f\|_{L^p(0,\infty)}, \qquad 1 < p < \infty. \tag{1.1.19}$$

Equivalently,

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p\,dx. \tag{1.1.20}$$

A generalized weighted Hardy inequality has the form

$$\left(\int_0^\infty |(Hf)(x)|^q u(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty |f(x)|^p v(x)\,dx\right)^{1/p}. \tag{1.1.21}$$

Here

$$u(x) \ge 0, \qquad v(x) \ge 0, \qquad 1 < p, q < \infty. \tag{1.1.22}$$

It is important at this point to find conditions for the weights u and v that will make the above inequality true.

The generalized Hardy operator is defined by

$$(H_\alpha f)(x) = x^{-\alpha}\int_0^x t^{\alpha-1} f(t)\,dt, \qquad \alpha > 0. \tag{1.1.23}$$

In multidimensional analysis, Hardy inequalities are written as

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p}\,dx \le \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx, \qquad 1 < p < n. \tag{1.1.24}$$

The weighted multidimensional version is

$$\int_\Omega |u(x)|^p w_1(x)\,dx \le C\int_\Omega |\nabla u(x)|^p w_2(x)\,dx. \tag{1.1.25}$$

One should note the importance of Muckenhoupt weights in weighted theory.

The class of Muckenhoupt weights A_p, where $1 < p < \infty$, consists of all positive weights w that satisfy the condition

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}}\,dx \right)^{p-1} < \infty. \tag{1.1.26}$$

In Orlicz spaces, modular functions are defined by

$$\rho_\Phi(f) = \int_0^\infty \Phi(|f(x)|)w(x)\,dx, \tag{1.1.27}$$

where Φ is a Young function.

Variable exponent spaces $L^{p(x)}$ are defined through

$$L^{p(x)}(\Omega) = \left\{ f : \int_\Omega |f(x)|^{p(x)}\,dx < \infty \right\}. \tag{1.1.28}$$

Their norm is

$$\|f\|_{L^{p(x)}} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \tag{1.1.29}$$

Hardy inequalities in variable exponent spaces are written as

$$\|Hf\|_{L^{q(x)}_u} \le C\|f\|_{L^{p(x)}_v}. \tag{1.1.30}$$

Fractional Hardy inequalities are expressed as

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\,dx \le C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy. \tag{1.1.31}$$

The best constant for Hardy-type inequalities is normally given by

$$C = \sup_{f \neq 0} \frac{\left(\int_0^\infty |(Hf)(x)|^q u(x)\,dx \right)^{1/q}}{\left(\int_0^\infty |f(x)|^p v(x)\,dx \right)^{1/p}}. \tag{1.1.32}$$

Such concepts serve as the foundation of research in weighted Hardy-type inequalities.

1.1.3 Improved Hardy Inequalities and Bessel Pairs

Recall the well-known Hardy inequality in $\mathbb{R}^n$, with $n \geq 3$:

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \;\geq\; \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} \, dx \tag{1.1.33}$$

here, every $u \in C_0^\infty(\mathbb{R}^n)$. Here, the constant $\left(\frac{n-2}{2}\right)^2$ is sharp. It is a well-known fact that the best constant in the above inequality cannot be achieved. Therefore, numerous studies have been conducted to refine Hardy type inequalities. One method would be to include more non-negative terms on the right hand side of the above inequality. But according to [20], Ghoussoub and Moradifam proved that there does not exist a strictly positive function $W \in C^1(0,\infty)$ such that

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \;\geq\; \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} \, dx \;+\; \int_{\mathbb{R}^n} W(|x|)\, |u|^2 dx$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$.

This scenario is different for bounded domains. Consider a bounded set $\Omega \subset \mathbb{R}^n$, where $n \geq 3$, and let $0 \in \Omega$. Brezis and Vázquez in [6] showed that for any inequality

$$\int_{\Omega} |\nabla u|^2 dx \;\geq\; \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} \, dx \;+\; z_0^2 \omega_n^{2/n} |\Omega|^{-2/n} \int_{\Omega} |u|^2 dx \tag{1.1.34}$$

is valid, where ω_n is the volume of the unit ball and $z_0 \approx 2.4048$ is the first root of the Bessel function $J_0(z)$. The quantity $z_0^2 \omega_n^{2/n} |\Omega|^{-2/n}$ is sharp if Ω is a ball. However, this inequality cannot be attained for functions in $W_0^{1,2}(\Omega)$. Brezis and Vázquez conjectured that there exist an infinite number of correction terms which one could add to the right-hand side of (2), where the term appearing in (2) is just the first of these terms. This issue has been addressed by numerous authors (see [2, 4, 13, 14, 15, 18, 23, 33, 42, 43], among others). For further background, the reader is referred to standard monographs such as [1, 22, 26, 27, 36].

Later, Ghoussoub and Moradifam [21, 22] provided a framework that extends and unifies many of these improvements. They proved the following theorem concerning Hardy-type inequalities:

Theorem A. Let $0 < R \leq \infty$. Suppose V, W are positive C^1-functions on $(0, R)$ satisfying

$$\int_0^R \frac{1}{r^{n-1} V(r)} \, dr = \infty \quad \text{and} \quad \int_0^R r^{n-1} V(r) \, dr < \infty.$$

Then the following statements are equivalent:

1. (V, W) forms an n-dimensional Bessel pair on $(0, R)$;

2. For all $u \in C_0^\infty(B_R)$, there holds

$$\int_{B_R} V(|x|)\,|\nabla u|^2 dx \;\geq\; \beta(V,W;R)\int_{B_R} W(|x|)|u|^2 dx,$$

where $\beta(V,W;R)$ is the best constant.

Here, a pair (V,W) of C^1-functions is said to be an n-dimensional Bessel pair on $(0,R)$ if there exists $c>0$ such that the ordinary differential equation

$$y''(r)+\left(\frac{n-1}{r}+\frac{V_r(r)}{V(r)}\right)y'(r)+\frac{cW(r)}{V(r)}y(r)=0$$

admits a positive solution on $(0,R)$.

Furthermore, $\beta(V,W;R)$ is defined as the supremum of all such c. It can also be shown that (V,W) forms an n-dimensional Bessel pair on $(0,R)$ if and only if $(r^{n-1}V, r^{n-1}W)$ is a one-dimensional Bessel pair on $(0,R)$. For further results and illustrative examples, see [21].

1.2 Preliminaries

Prior to the main results presented in this dissertation, we first recall the setting of the Baouendi–Grushin problem and its relation to Hardy inequalities, which are relevant to the analysis. We consider the sub-Riemannian space

$$\mathbb{R}^{m+k}=\{z=(x,y): x\in\mathbb{R}^m,\, y\in\mathbb{R}^k\}$$

defined by the Baouendi–Grushin vector fields

$$X_i=\frac{\partial}{\partial x_i},\quad i=1,\dots,m,\qquad Y_j=|x|^\gamma\frac{\partial}{\partial y_j},\quad j=1,\dots,k. \tag{1.2.1}$$

In the above expression, $\gamma\geq 0$ is a non-negative real number, and $|x|$ denotes the standard Euclidean norm of $x\in\mathbb{R}^m$.Hardy-type inequalities related to Grushin type operators were studied by D'Ambrosio [**?**]. In particular, one has the inequality

$$\int_{\mathbb{R}^n}\left(|\nabla_x f|^2+|x|^{2\gamma}|\nabla_y f|^2\right)dz\geq\left(\frac{Q-2}{2}\right)^2\int_{\mathbb{R}^n}\frac{|x|^{2\gamma}}{|x|^{2\gamma+2}+(1+\gamma)^2|y|^2}|f|^2dz, \tag{1.4}$$

where $n=m+k$, $Q=m+(1+\gamma)k$ and $f\in C_0^\infty(\mathbb{R}^m\times\mathbb{R}^k\setminus\{(0,0)\})$. $\nabla_x f$ and $\nabla_y f$ denote the gradients of f with respect to the variables x and y, respectively. In the special case of $\gamma=0$, the inequality (1.4) reduces to the classical multidimensional L^2-Hardy inequality (i.e. $p=2$ of (1.1)).

Now we give some additional facts on the Baouendi-Grushin framework. First of all, the corresponding gradient given by the vector fields (1.3) is characterized

by the following

$$\nabla_\gamma = (X_1, \ldots, X_m, Y_1, \ldots, Y_k) = (\nabla_x, |x|^\gamma \nabla_y).$$

The Baouendi–Grushin operator is defined as a differential operator on $\mathbb{R}^{m+k}$ by

$$\Delta_\gamma = \sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{k} Y_j^2 = \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma.$$

The operator in itself has many interesting properties. For example, it is not uniformly elliptic in the space $\mathbb{R}^{m+k}$ (due to the degeneracy on the subspace $\{0\} \times \mathbb{R}^k$). It is also sub-elliptic and for non-negative even integers γ satisfies the Hörmander condition, meaning it can be expressed as a sum of squares of vector fields that form a Lie algebra of full rank at any point in $\mathbb{R}^{m+k}$. We also have the anisotropic dilation defined by

$$\delta_a(x, y) = (ax, a^{1+\gamma} y),$$

where $a > 0$. The homogeneous dimension with respect to the dilation is

$$Q = m + (1 + \gamma)k.$$

Here we expand the scalar product $\nabla_\gamma \rho \cdot \nabla_\gamma u$ using the definition of the Baouendi–Grushin gradient $\nabla_\gamma = (\nabla_x, |x|^\gamma \nabla_y)$. The corresponding distance function $\rho(z)$ is given by

$$\rho = \rho(z) = \left(|x|^{2(1+\gamma)} + (1+\gamma)^2 |y|^2 \right)^{\frac{1}{2(1+\gamma)}}.$$

$$|x| = \sqrt{x_1^2 + \cdots + x_m^2},$$

$$|y| = \sqrt{y_1^2 + \cdots + y_k^2}$$

A direct computation gives the following formulas:

$$\nabla_\gamma \rho = \left(\frac{|x|^{2\gamma} x}{\rho^{2\gamma+1}}, \frac{(1+\gamma)|x|^\gamma y}{\rho^{2\gamma+1}} \right)$$

and

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}.$$

Chapter 2

Weighted Hardy inequalities and identities related to the Baouendi-Grushin operator

This chapter is devoted to a new generalized form of weighted Hardy-type identities in the Grushin-type setting. More precisely, by expressing the radial derivative in the form

$$\frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|}$$

and employing the notion of a Bessel pair, we establish an improved Hardy identity. Such identities provide a strengthened version of the classical Hardy inequality, where not only an estimate is obtained, but also an exact remainder term is explicitly identified.

Since Grushin-type operators possess an anisotropic and degenerate structure, the corresponding functional inequalities differ substantially from those in the Euclidean setting. Therefore, the proper choice of the radial function ρ, together with the properties of $\nabla_\gamma \rho$ and $|\nabla_\gamma \rho|$, as well as the introduction of weights associated with a Bessel pair, forms the main methodological basis of this chapter.

The principal aim of this chapter is to prove an improved Hardy identity for the weighted radial Grushin derivative. More specifically, under the assumption that the weight functions V and W form a Bessel pair, we show that the difference

$$\int_{\mathbb{R}^n} V(\rho) \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 W(\rho) \, |u|^2 \, dz \geq 0.$$

can be represented as a non-negative quadratic remainder term. This result yields not merely a Hardy-type inequality, but rather its precise structural identity.

The proof in this chapter is based on the factorization method for first-order differential operators. This approach makes it possible to compute the formal adjoint of the operator, to expand the composition T^*T, and to simplify the zero-order term by means of the Bessel pair equation. As a consequence, the resulting

identity directly implies a strengthened form of the Hardy-type inequality.

With this preparation, we now state and prove the main result of this chapter, namely, the improved Hardy identity with more general weights.

2.1 The improved Hardy identity with more general weights

The following theorem generalizes the corresponding weighted Hardy-type identities from the classical Euclidean radial setting to the Grushin-type structure. In the case $\gamma = 0$, it reduces to the Bessel pair identity studied by N. T. Duy, N. Lam, and N. Triet in [12].

Theorem 1 (Improved Hardy identity with more general weights)**.** *Let $0 < R \leq \infty$, V and W be positive C^1-functions on $(0, R)$. Assume that $\left(\rho^{Q-1}V, \rho^{Q-1}W\right)$ is a Bessel pair on $(0, R)$. Then for $u \in C_0^\infty(B_R \backslash \{0\})$:*

$$\begin{aligned} &\int_{\mathbb{R}^n} V(\rho) \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 W(\rho) |u|^2 \, dz \\ &= \int_{\mathbb{R}^n} V(\rho)\varphi^2(\rho) \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma}{|\nabla_\gamma \rho|} \left(\frac{u}{\varphi(\rho)} \right) \right|^2 dz \end{aligned} \tag{2.1.1}$$

Here φ is the positive solution of

$$\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)' + \rho^{Q-1}W(\rho)\varphi(\rho) = 0 \tag{2.1.2}$$

on the interval $(0, R)$.

Here B_R denotes the ball with respect to the Grushin distance function ρ, namely

$$B_R = \{z \in \mathbb{R}^{m+k} : \rho(z) < R\}.$$

Proof. We define the first-order differential operator T by

$$Tu = \sqrt{V(\rho)} \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} - \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| u.$$

First, we compute the formal adjoint of the operator T. By definition, T^* satisfies

$$\int_{\mathbb{R}^n} Tu \, \overline{v} \, dz = \int_{\mathbb{R}^n} u \, \overline{T^* v} \, dz.$$

for all test functions $u, v \in C_0^\infty(B_R \setminus \{0\})$. For this purpose, we consider the inner product of Tu with a test function v and apply integration by parts.

$$\int_{\mathbb{R}^n} Tu\overline{v} dz = \int_{\mathbb{R}^n} \sqrt{V(\rho)} \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \overline{v} dz - \int_{\mathbb{R}^n} \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| u\overline{v} dz.$$

Using the explicit form of the Grushin gradient, we rewrite the scalar product $\nabla_\gamma \rho \cdot \nabla_\gamma u$ in terms of the x and y variables.

$$\int_{\mathbb{R}^n} Tu\overline{v}dz = \int_{\mathbb{R}^n} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left(\sum_{i=1}^{m} \partial_{x_i}\rho\, \partial_{x_i} u + |x|^{2\gamma} \sum_{j=1}^{k} \partial_{y_j}\rho\, \partial_{y_j} u \right) \overline{v}\, dz$$

$$- \int_{\mathbb{R}^n} \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| u\, \overline{v}\, dz.$$

Next, we apply integration by parts to the terms containing derivatives of u. This allows us to transfer the derivatives from u to the test function v. Expanding the expression, we get

$$\int_{\mathbb{R}^n} Tu\overline{v}dz = -\sum_{i=1}^{m} \int_{\mathbb{R}^n} \partial_{x_i} \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \partial_{x_i}\rho\, u\overline{v}\, dz - \sum_{i=1}^{m} \int_{\mathbb{R}^n} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \partial_{x_i}^2 \rho\, u\overline{v}\, dz$$

$$- \sum_{i=1}^{m} \int_{\mathbb{R}^n} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \partial_{x_i}\rho\, u\, \overline{\partial_{x_i} v}\, dz - \sum_{j=1}^{k} \int_{\mathbb{R}^n} |x|^{2\gamma} \partial_{y_j} \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \partial_{y_j}\rho\, u\overline{v}\, dz$$

$$- \sum_{j=1}^{k} \int_{\mathbb{R}^n} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} |x|^{2\gamma} \partial_{y_j}^2 \rho\, u\overline{v}\, dz - \sum_{j=1}^{k} \int_{\mathbb{R}^n} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} |x|^{2\gamma} \partial_{y_j}\rho\, u\, \overline{\partial_{y_j} v}\, dz$$

$$- \int_{\mathbb{R}^n} \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| u\, \overline{v}\, dz = - \int_{\mathbb{R}^n} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left(\sum_{i=1}^{m} \partial_{x_i}^2 \rho + |x|^{2\gamma} \sum_{j=1}^{k} \partial_{y_j}^2 \rho \right) u\overline{v}\, dz$$

$$- \int_{\mathbb{R}^n} \left(\sum_{i=1}^{m} \partial_{x_i} \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \partial_{x_i}\rho + |x|^{2\gamma} \sum_{j=1}^{k} \partial_{y_j} \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \partial_{y_j}\rho \right) u\overline{v}\, dz$$

$$- \int_{\mathbb{R}^n} \left(\sum_{i=1}^{m} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \partial_{x_i}\rho\, \overline{\partial_{x_i} v} + |x|^{2\gamma} \sum_{j=1}^{k} \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \partial_{y_j}\rho\, \overline{\partial_{y_j} v} \right) u\, dz$$

$$- \int_{\mathbb{R}^n} \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| u\, \overline{v}\, dz.$$

Therefore, we obtain

$$T^*v = -\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma \rho \cdot \nabla_\gamma v\right) - \left[\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\right)\cdot \nabla_\gamma \rho\right] v$$
$$- \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\Delta_\gamma \rho\, v - \sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho| v.$$

Using the identity

$$\Delta_\gamma \rho = \frac{Q-1}{\rho}|\nabla_\gamma \rho|^2,$$

we get

$$-\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\Delta_\gamma \rho\, v = -\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\frac{Q-1}{\rho}|\nabla_\gamma \rho|^2 v = -\sqrt{V(\rho)}\frac{Q-1}{\rho}|\nabla_\gamma \rho| v.$$

Hence,

$$T^*v = -\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma \rho \cdot \nabla_\gamma v\right) - \left[\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\right)\cdot \nabla_\gamma \rho\right] v$$
$$- \sqrt{V(\rho)}\frac{Q-1}{\rho}|\nabla_\gamma \rho| v - \sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho| v.$$

Equivalently, the formal adjoint operator T^* is given by

$$T^* = -\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma \rho \cdot \nabla_\gamma\right) - \left[\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\right)\cdot \nabla_\gamma \rho\right]$$
$$- \sqrt{V(\rho)}\frac{Q-1}{\rho}|\nabla_\gamma \rho| - \sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho|.$$

After finding the formal adjoint T^*, we compute the composition T^*Tu. This computation is needed to derive the main Hardy-type identity.

$$T^*Tu = T^*(Tu) = -\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left[\nabla_\gamma \rho \cdot \nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right) - \sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho| u\right)\right]$$

$$- \left[\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\right)\cdot \nabla_\gamma \rho\right]\left[\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right) - \sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho| u\right]$$

$$-\sqrt{V(\rho)}\frac{Q-1}{\rho}|\nabla_\gamma \rho|\left[\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right) - \sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho| u\right]$$

$$-\sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho|\left[\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right) - \sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma \rho| u\right]$$

To simplify the expression for T^*Tu, we first compute the gradient of the expression inside the brackets. By the product rule, this derivative is decomposed into four terms.

$$\nabla_\gamma \left[\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right) - \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| u \right]$$

$$= \nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right) + \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)$$

$$-\nabla_\gamma \left(\sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \right) u - \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \nabla_\gamma u$$

Thus, T^*Tu can be represented as the sum of the following terms:

$$T^*Tu = I_1 + I_2 + \cdots + I_{10}$$

Here, the terms $I_1, \ldots, I_{10}$ are obtained by expanding the product T^*Tu term by term.

$$I_1 = -\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \left[\nabla_\gamma \rho \cdot \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)\right]$$

$$I_2 = -\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left[\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right] \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)$$

$$I_3 = \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left[\nabla_\gamma \left(\sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \right) \cdot \nabla_\gamma \rho \right] u$$

$$I_4 = V(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)$$

$$I_5 = -\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left[\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right] \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)$$

$$I_6 = \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \left[\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right] u$$

$$I_7 = -V(\rho) \frac{Q-1}{\rho} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)$$

$$I_8 = V(\rho) \frac{Q-1}{\rho} |\nabla_\gamma \rho|^2 \frac{\varphi'(\rho)}{\varphi(\rho)} u$$

$$I_9 = -V(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)$$

$$I_{10} = V(\rho) \left(\frac{\varphi'(\rho)}{\varphi(\rho)} \right)^2 |\nabla_\gamma \rho|^2 u$$

Here, the terms $I_1, \ldots, I_{10}$ are obtained by expanding the product T^*Tu term by term.

$$I_4 + I_9 = 0$$

Therefore, after this cancellation, T^*Tu is reduced to the remaining eight terms.

$$T^*Tu = I_1 + I_2 + I_3 + I_5 + I_6 + I_7 + I_8 + I_{10}$$

Combining the similar terms I_2 and I_5, we obtain the following compact form.

$$T^*Tu = -\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \left[\nabla_\gamma \rho \cdot \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \right]$$

$$-2\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left[\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right] \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right)$$

$$-V(\rho)\frac{Q-1}{\rho} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) + Z$$

Here, Z denotes the sum of the zero-order terms, that is, the terms containing u without derivatives.

$$Z = \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left[\nabla_\gamma \left(\sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \right) \cdot \nabla_\gamma \rho \right] u$$

$$+\sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \left[\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right] u +$$

$$V(\rho)\frac{Q-1}{\rho} |\nabla_\gamma \rho|^2 \frac{\varphi'(\rho)}{\varphi(\rho)} u + V(\rho) \left(\frac{\varphi'(\rho)}{\varphi(\rho)} \right)^2 |\nabla_\gamma \rho|^2 u$$

To simplify Z, we first compute the derivative of the coefficient containing φ.

$$\nabla_\gamma \left(\sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \right) = \frac{V'(\rho)}{2\sqrt{V(\rho)}} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \nabla_\gamma \rho +$$

$$\sqrt{V(\rho)} \left[\frac{\varphi''(\rho)}{\varphi(\rho)} - \left(\frac{\varphi'(\rho)}{\varphi(\rho)} \right)^2 \right] |\nabla_\gamma \rho| \nabla_\gamma \rho + \sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} \nabla_\gamma |\nabla_\gamma \rho|$$

Multiplying this expression by $\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}$ and taking the scalar product with $\nabla_\gamma \rho$, we get

$$\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left[\nabla_\gamma \left(\sqrt{V(\rho)} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho| \right) \cdot \nabla_\gamma \rho \right] = \frac{V'(\rho)}{2} \frac{\varphi'(\rho)}{\varphi(\rho)} |\nabla_\gamma \rho|^2 +$$

$$V(\rho)\left[\frac{\varphi''(\rho)}{\varphi(\rho)}-\left(\frac{\varphi'(\rho)}{\varphi(\rho)}\right)^2\right]|\nabla_\gamma\rho|^2+V(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{\nabla_\gamma|\nabla_\gamma\rho|\cdot\nabla_\gamma\rho}{|\nabla_\gamma\rho|}$$

Next, we compute the derivative of the coefficient $\frac{\sqrt{V(\rho)}}{|\nabla_\gamma\rho|}$.

$$\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma\rho|}\right)=\frac{V'(\rho)}{2\sqrt{V(\rho)}|\nabla_\gamma\rho|}\nabla_\gamma\rho-\frac{\sqrt{V(\rho)}}{|\nabla_\gamma\rho|^2}\nabla_\gamma|\nabla_\gamma\rho|$$

Substituting this derivative into the second zero-order term gives

$$\sqrt{V(\rho)}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma\rho|\left[\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma\rho|}\right)\cdot\nabla_\gamma\rho\right]$$

$$=\frac{V'(\rho)}{2}\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma\rho|^2-V(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{\nabla_\gamma|\nabla_\gamma\rho|\cdot\nabla_\gamma\rho}{|\nabla_\gamma\rho|}$$

Combining the obtained expressions, we find the following form of Z.

$$Z=V(\rho)\frac{\varphi''(\rho)}{\varphi(\rho)}|\nabla_\gamma\rho|^2u-V(\rho)\left(\frac{\varphi'(\rho)}{\varphi(\rho)}\right)^2|\nabla_\gamma\rho|^2u$$

$$+V'(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma\rho|^2u+V(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{\nabla_\gamma|\nabla_\gamma\rho|\cdot\nabla_\gamma\rho}{|\nabla_\gamma\rho|}u$$

$$-V(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{\nabla_\gamma|\nabla_\gamma\rho|\cdot\nabla_\gamma\rho}{|\nabla_\gamma\rho|}u+V(\rho)\frac{Q-1}{\rho}|\nabla_\gamma\rho|^2\frac{\varphi'(\rho)}{\varphi(\rho)}u+V(\rho)\left(\frac{\varphi'(\rho)}{\varphi(\rho)}\right)^2|\nabla_\gamma\rho|^2u$$

The terms containing $\nabla_\gamma|\nabla_\gamma\rho|\cdot\nabla_\gamma\rho$ cancel each other.

$$V(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{\nabla_\gamma|\nabla_\gamma\rho|\cdot\nabla_\gamma\rho}{|\nabla_\gamma\rho|}-V(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{\nabla_\gamma|\nabla_\gamma\rho|\cdot\nabla_\gamma\rho}{|\nabla_\gamma\rho|}=0$$

Similarly, the terms containing $\left(\frac{\varphi'(\rho)}{\varphi(\rho)}\right)^2$ also cancel each other

$$-V(\rho)\left(\frac{\varphi'(\rho)}{\varphi(\rho)}\right)^2|\nabla_\gamma\rho|^2+V(\rho)\left(\frac{\varphi'(\rho)}{\varphi(\rho)}\right)^2|\nabla_\gamma\rho|^2=0$$

Thus, only the following terms remain.

$$Z=|\nabla_\gamma\rho|^2\left[V(\rho)\frac{\varphi''(\rho)}{\varphi(\rho)}+V'(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}+V(\rho)\frac{Q-1}{\rho}\frac{\varphi'(\rho)}{\varphi(\rho)}\right]u$$

Using the Bessel pair equation, we simplify the zero-order coefficient Z.

$$\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'+\rho^{Q-1}W(\rho)\varphi(\rho)=0$$

Expanding the derivative in the first term, we obtain the following equivalent form.

$$V(\rho)\varphi''(\rho) + V'(\rho)\varphi'(\rho) + V(\rho)\frac{Q-1}{\rho}\varphi'(\rho) + W(\rho)\varphi(\rho) = 0$$

$$V(\rho)\frac{\varphi''(\rho)}{\varphi(\rho)} + V'(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)} + V(\rho)\frac{Q-1}{\rho}\frac{\varphi'(\rho)}{\varphi(\rho)} = -W(\rho)$$

Therefore, the zero-order coefficient becomes

$$Z = -|\nabla_\gamma \rho|^2 W(\rho)u$$

Substituting this expression into the formula for T^*Tu, we obtain the following representation.

$$T^*Tu = -\frac{V(\rho)}{|\nabla_\gamma \rho|^2}\left[\nabla_\gamma \rho \cdot \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)\right]$$

$$-\left[2\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\right)\cdot \nabla_\gamma \rho\right) + V(\rho)\frac{Q-1}{\rho}\right](\nabla_\gamma \rho \cdot \nabla_\gamma u)$$

$$-|\nabla_\gamma \rho|^2 W(\rho)u$$

Next, we multiply both sides by $\overline{u}$ and integrate over $\mathbb{R}^n$.

$$\int_{\mathbb{R}^n}(T^*Tu)\overline{u}\,dz = -\int_{\mathbb{R}^n}\frac{V(\rho)}{|\nabla_\gamma \rho|^2}\left[\nabla_\gamma \rho \cdot \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)\right]\overline{u}\,dz$$

$$-\int_{\mathbb{R}^n}\left[2\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\left(\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|}\right)\cdot \nabla_\gamma \rho\right) + V(\rho)\frac{Q-1}{\rho}\right](\nabla_\gamma \rho \cdot \nabla_\gamma u)\,\overline{u}\,dz$$

$$-\int_{\mathbb{R}^n}|\nabla_\gamma \rho|^2 W(\rho)|u|^2\,dz$$

Now, if we compute the first term, we obtain

$$-\int\frac{V(\rho)}{|\nabla_\gamma \rho|^2}\left[\nabla_\gamma \rho \cdot \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)\right]\overline{u}\,dz$$

$$=\int_{\mathbb{R}^n}(\nabla_\gamma \rho \cdot \nabla_\gamma u)\left[\nabla_\gamma \rho \cdot \nabla_\gamma\left(\frac{V(\rho)}{|\nabla_\gamma \rho|^2}\overline{u}\right)\right]dz$$

$$+\int_{\mathbb{R}^n}\frac{V(\rho)}{|\nabla_\gamma \rho|^2}\frac{Q-1}{\rho}|\nabla_\gamma \rho|^2\left(\nabla_\gamma \rho \cdot \nabla_\gamma u\right)\overline{u}\,dz$$

Using the product rule, we expand the derivative in the first term.

$$\nabla_\gamma \left(\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \overline{u} \right) = \nabla_\gamma \left(\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \right) \overline{u} + \frac{V(\rho)}{|\nabla_\gamma \rho|^2} \nabla_\gamma \overline{u}$$

Taking the scalar product with $\nabla_\gamma \rho$, we obtain

$$\nabla_\gamma \rho \cdot \nabla_\gamma \left(\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \overline{u} \right) = \left[\nabla_\gamma \left(\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \right) \cdot \nabla_\gamma \rho \right] \overline{u} + \frac{V(\rho)}{|\nabla_\gamma \rho|^2} \left(\nabla_\gamma \rho \cdot \nabla_\gamma \overline{u} \right)$$

Substituting this expression into the previous identity gives

$$- \int_{\mathbb{R}^n} \frac{V(\rho)}{|\nabla_\gamma \rho|^2} \left[\nabla_\gamma \rho \cdot \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \right] \overline{u} \, dz$$

$$= \int_{\mathbb{R}^n} \left[\nabla_\gamma \left(\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \right) \cdot \nabla_\gamma \rho \right] \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \overline{u} \, dz$$

$$+ \int_{\mathbb{R}^n} \frac{V(\rho)}{|\nabla_\gamma \rho|^2} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \left(\nabla_\gamma \rho \cdot \nabla_\gamma \overline{u} \right) dz$$

$$+ \int_{\mathbb{R}^n} V(\rho) \frac{Q-1}{\rho} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \overline{u} \, dz$$

Using the following identity, the coefficient of the mixed term can be rewritten in a convenient form.

$$\nabla_\gamma \left(\frac{V(\rho)}{|\nabla_\gamma \rho|^2} \right) \cdot \nabla_\gamma \rho = 2 \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left(\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right)$$

we obtain

$$- \int_{\mathbb{R}^n} \frac{V(\rho)}{|\nabla_\gamma \rho|^2} \left[\nabla_\gamma \rho \cdot \nabla_\gamma \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \right] \overline{u} \, dz$$

$$= \int_{\mathbb{R}^n} \left[2 \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left(\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right) + V(\rho) \frac{Q-1}{\rho} \right] \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \overline{u} \, dz$$

$$+ \int_{\mathbb{R}^n} \frac{V(\rho)}{|\nabla_\gamma \rho|^2} \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \left(\nabla_\gamma \rho \cdot \nabla_\gamma \overline{u} \right) dz$$

Substituting this back into the integral identity, the mixed terms appear with opposite signs.

$$\int_{\mathbb{R}^n} (T^* T u) \overline{u} \, dz = \int_{\mathbb{R}^n} \left[2 \frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \left(\nabla_\gamma \left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma \rho|} \right) \cdot \nabla_\gamma \rho \right) + V(\rho) \frac{Q-1}{\rho} \right] \left(\nabla_\gamma \rho \cdot \nabla_\gamma u \right) \overline{u} \, dz$$

$$+\int_{\mathbb{R}^n}\frac{V(\rho)}{|\nabla_\gamma\rho|^2}\left(\nabla_\gamma\rho\cdot\nabla_\gamma u\right)\left(\nabla_\gamma\rho\cdot\nabla_\gamma\overline{u}\right)dz$$

$$-\int_{\mathbb{R}^n}\left[2\frac{\sqrt{V(\rho)}}{|\nabla_\gamma\rho|}\left(\nabla_\gamma\left(\frac{\sqrt{V(\rho)}}{|\nabla_\gamma\rho|}\right)\cdot\nabla_\gamma\rho\right)+V(\rho)\frac{Q-1}{\rho}\right]\left(\nabla_\gamma\rho\cdot\nabla_\gamma u\right)\overline{u}\,dz$$

$$-\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2W(\rho)|u|^2\,dz$$

The mixed terms cancel each other. Hence, we arrive at the following identity.

$$\int_{\mathbb{R}^n}(T^*Tu)\overline{u}\,dz=\int_{\mathbb{R}^n}\frac{V(\rho)}{|\nabla_\gamma\rho|^2}\left(\nabla_\gamma\rho\cdot\nabla_\gamma u\right)\left(\nabla_\gamma\rho\cdot\nabla_\gamma\overline{u}\right)dz-\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2W(\rho)|u|^2\,dz$$

Since the second factor is the complex conjugate of the first one, their product is equal to the squared modulus.

$$\left(\nabla_\gamma\rho\cdot\nabla_\gamma u\right)\left(\nabla_\gamma\rho\cdot\nabla_\gamma\overline{u}\right)=\left|\nabla_\gamma\rho\cdot\nabla_\gamma u\right|^2$$

Therefore, the left-hand side can be written as

$$\int_{\mathbb{R}^n}(T^*Tu)\overline{u}\,dz=\int_{\mathbb{R}^n}\frac{V(\rho)}{|\nabla_\gamma\rho|^2}\left|\nabla_\gamma\rho\cdot\nabla_\gamma u\right|^2dz-\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2W(\rho)|u|^2\,dz$$

It remains to rewrite the right-hand side in factorized form.

$$\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\frac{u}{\varphi(\rho)}\right)=\frac{1}{\varphi(\rho)}\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}-\frac{u}{\varphi^2(\rho)}\frac{\nabla_\gamma\rho\cdot\nabla_\gamma(\varphi(\rho))}{|\nabla_\gamma\rho|}$$

By the chain rule, we have

$$\nabla_\gamma(\varphi(\rho))=\varphi'(\rho)\nabla_\gamma\rho$$

Consequently,

$$\frac{\nabla_\gamma\rho\cdot\nabla_\gamma(\varphi(\rho))}{|\nabla_\gamma\rho|}=\varphi'(\rho)\frac{\nabla_\gamma\rho\cdot\nabla_\gamma\rho}{|\nabla_\gamma\rho|}=\varphi'(\rho)|\nabla_\gamma\rho|$$

Substituting this into the previous expression, we get

$$\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\frac{u}{\varphi(\rho)}\right)=\frac{1}{\varphi(\rho)}\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}-\frac{\varphi'(\rho)}{\varphi^2(\rho)}|\nabla_\gamma\rho|u$$

Multiplying both sides by $\varphi(\rho)$, we obtain

$$\varphi(\rho)\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\frac{u}{\varphi(\rho)}\right)=\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}-\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma\rho|u$$

After multiplying by $V(\rho)$ and taking the squared modulus, we obtain the factorized expression.

$$V(\rho)\varphi^2(\rho)\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\frac{u}{\varphi(\rho)}\right)\right|^2 = V(\rho)\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}-\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma\rho|u\right|^2$$

Integrating both sides over $\mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n} V(\rho)\varphi^2(\rho)\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\frac{u}{\varphi(\rho)}\right)\right|^2 dz = \int_{\mathbb{R}^n} V(\rho)\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}-\frac{\varphi'(\rho)}{\varphi(\rho)}|\nabla_\gamma\rho|u\right|^2 dz$$

Combining the previous identities, we finally obtain the improved Hardy-type identity

$$\int_{\mathbb{R}^n}(T^*Tu)\overline{u}\,dz = \int_{\mathbb{R}^n}|Tu|^2\,dz$$

$$\int_{\mathbb{R}^n} V(\rho)\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}\right|^2 dz - \int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2 W(\rho)|u|^2\,dz$$

$$= \int_{\mathbb{R}^n} V(\rho)\varphi^2(\rho)\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\frac{u}{\varphi(\rho)}\right)\right|^2 dz$$

□

Remark 2.1.1. In the work of Nguyen Tuan Duy and Le Long Phi, "Cylindrical Hardy Type Inequalities with Bessel Pairs", the authors obtained weighted Hardy type identities in a cylindrical setting [12]. Their result is mainly based on the Euclidean radial variable $|y|$, the usual gradient ∇u, and the radial derivative in the direction $\frac{y}{|y|}\cdot\nabla_y u$.

In this case, the weights V and W depend on the cylindrical variable $|y|$, and the Bessel pair condition is written in the form $(r^{k-1}V, r^{k-1}W)$.

Our theorem extends this idea to a more general Grushin-type structure. In our result, the Euclidean radial variable $|y|$ is replaced by the homogeneous quasi-norm ρ, and the usual gradient is replaced by the Grushin gradient ∇_γ. Therefore, the ordinary radial derivative is replaced by the Grushin-type radial derivative $\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}$.

This is the main difference between the two results. The theorem of Nguyen Tuan Duy and Le Long Phi is formulated in a cylindrical Euclidean framework, while our theorem is formulated in a degenerate Grushin-type geometric framework. Moreover, the Bessel pair condition in our theorem is adapted to the homogeneous dimension Q, which makes the result applicable to a wider class of weighted Hardy identities.

Thus, our result can be regarded as a generalization of the weighted Hardy identity obtained by Nguyen Tuan Duy and Le Long Phi. It preserves the main idea of the Bessel pair method, but extends it from the cylindrical Euclidean setting to the Grushin-type setting with a more general radial structure and a

more general weighted form.

2.2 Applications and Special Cases of the Main Theorem

In this subsection, we derive several consequences of the main theorem. These results show that the improved Hardy identity provides a unified framework for obtaining weighted Hardy-type identities in the Grushin setting.

In particular, when $\gamma = 0$, the Baouendi–Grushin structure reduces to the classical Euclidean setting. Indeed, in this case

$$\nabla_\gamma = (\nabla_x, \nabla_y) = \nabla, \qquad Q = m + k = n,$$

and the Grushin radial function becomes

$$\rho(z) = \left(|x|^2 + |y|^2\right)^{\frac{1}{2}} = |z|.$$

Hence,

$$|\nabla_\gamma \rho| = |\nabla |z|| = 1.$$

Therefore, our result reduces to the radial Bessel pair identity obtained by N. T. Duy, N. Lam, and N. Triet in [12].

Substituting $\gamma = 0$, $Q = n$, $\rho = |z|$, and $|\nabla_\gamma \rho| = 1$ into our result, we obtain

$$\int_{\mathbb{R}^n} V(|z|)\,|\nabla|z| \cdot \nabla u|^2\,dz - \int_{\mathbb{R}^n} W(|z|)|u|^2 dz = \int_{\mathbb{R}^n} V(|z|)\varphi^2(|z|)\left|\nabla|z| \cdot \nabla\left(\frac{u}{\varphi(|z|)}\right)\right|^2 dz.$$

Since

$$\nabla|z| \cdot \nabla u = \frac{du}{d|z|},$$

this identity can be written as

$$\int_{\mathbb{R}^n} V(|z|)\left|\frac{du}{d|z|}\right|^2 dz - \int_{\mathbb{R}^n} W(|z|)|u|^2 dz = \int_{\mathbb{R}^n} V(|z|)\varphi^2(|z|)\left|\frac{d}{d|z|}\left(\frac{u}{\varphi}\right)\right|^2 dz.$$

Thus, in the case $\gamma = 0$, our result gives the radial Bessel pair identity in the Euclidean setting.

The corresponding Bessel pair equation also reduces to the classical radial form. Since $Q = n$, we have

$$\left(r^{Q-1}V(r)\varphi'(r)\right)' + r^{Q-1}W(r)\varphi(r) = 0,$$

and therefore

$$\left(r^{n-1}V(r)\varphi'(r)\right)' + r^{n-1}W(r)\varphi(r) = 0.$$

Dividing by $r^{n-1}V(r)$, we obtain

$$\varphi''(r) + \left(\frac{n-1}{r} + \frac{V'(r)}{V(r)}\right)\varphi'(r) + \frac{W(r)}{V(r)}\varphi(r) = 0.$$

Hence, the case $\gamma = 0$ of our result coincides with the radial Bessel pair identity known in the Euclidean framework.

Special case 2.2.1 (Power-weighted Grushin-Hardy identity)**.** In this special case, we choose the weight function in the form

$$V(\rho) = \rho^{-\alpha},$$

where $\alpha \in \mathbb{R}$. We also take

$$u \in C_0^\infty(B_R \backslash \{0\})$$

and choose the positive function

$$\varphi(\rho) = \rho^{-\frac{Q-2-\alpha}{2}}.$$

From formula (2.1.2), we have

$$\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)' + \rho^{Q-1}W(\rho)\varphi(\rho) = 0.$$

Hence,

$$\rho^{Q-1}W(\rho)\varphi(\rho) = -\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'.$$

Therefore,

$$W(\rho) = -\frac{\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'}{\rho^{Q-1}\varphi(\rho)}.$$

Now we compute $W(\rho)$ for the above choice of V and φ. Since

$$V(\rho) = \rho^{-\alpha}, \qquad \varphi(\rho) = \rho^{-\frac{Q-2-\alpha}{2}}.$$

by the power rule for differentiation, we get

$$\varphi'(\rho) = -\frac{Q-2-\alpha}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}.$$

Then

$$\begin{aligned}
\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)' &= \left(\rho^{Q-1}\rho^{-\alpha}\left(-\frac{Q-2-\alpha}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\right)\right)' \\
&= \left(-\frac{Q-2-\alpha}{2}\rho^{Q-1-\alpha-\frac{Q-2-\alpha}{2}-1}\right)' \\
&= \left(-\frac{Q-2-\alpha}{2}\rho^{Q-\alpha-2-\frac{Q-2-\alpha}{2}}\right)' \\
&= \left(-\frac{Q-2-\alpha}{2}\rho^{\frac{Q-2-\alpha}{2}}\right)' \\
&= -\frac{Q-2-\alpha}{2}\cdot\frac{Q-2-\alpha}{2}\rho^{\frac{Q-2-\alpha}{2}-1} \\
&= -\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1}.
\end{aligned}$$

Substituting this expression into the formula for $W(\rho)$, we obtain

$$\begin{aligned}
W(\rho) &= -\frac{-\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1}}{\rho^{Q-1}\rho^{-\frac{Q-2-\alpha}{2}}} \\
&= \left(\frac{Q-2-\alpha}{2}\right)^2\frac{\rho^{\frac{Q-2-\alpha}{2}-1}}{\rho^{Q-1-\frac{Q-2-\alpha}{2}}} \\
&= \left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1-\left(Q-1-\frac{Q-2-\alpha}{2}\right)} \\
&= \left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1-Q+1+\frac{Q-2-\alpha}{2}} \\
&= \left(\frac{Q-2-\alpha}{2}\right)^2\rho^{Q-2-\alpha-Q} \\
&= \left(\frac{Q-2-\alpha}{2}\right)^2\rho^{-2-\alpha}.
\end{aligned}$$

Thus, the corresponding potential is

$$W(\rho) = \left(\frac{Q-2-\alpha}{2}\right)^2\rho^{-2-\alpha}.$$

Now we substitute

$$V(\rho) = \rho^{-\alpha}, \qquad \varphi(\rho) = \rho^{-\frac{Q-2-\alpha}{2}}, \qquad W(\rho) = \left(\frac{Q-2-\alpha}{2}\right)^2\rho^{-2-\alpha}$$

into our main identity.

Then the left-hand side becomes

$$LHS = \int_{\mathbb{R}^n} \rho^{-\alpha} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 W(\rho) |u|^2 \, dz$$
$$= \int_{\mathbb{R}^n} \rho^{-\alpha} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 \left[\left(\frac{Q-2-\alpha}{2} \right)^2 \rho^{-2-\alpha} \right] |u|^2 \, dz.$$

Now we calculate the right-hand side of the main identity. By the choice of φ, we have

$$\varphi^2(\rho) = \left(\rho^{-\frac{Q-2-\alpha}{2}} \right)^2 = \rho^{-(Q-2-\alpha)}.$$

Hence,

$$V(\rho)\varphi^2(\rho) = \rho^{-\alpha} \rho^{-(Q-2-\alpha)} = \rho^{-\alpha-Q+2+\alpha} = \rho^{-(Q-2)}.$$

Also,

$$\frac{u}{\varphi(\rho)} = \frac{u}{\rho^{-\frac{Q-2-\alpha}{2}}} = \rho^{\frac{Q-2-\alpha}{2}} u.$$

Therefore, the right-hand side becomes

$$RHS = \int_{\mathbb{R}^n} V(\rho)\varphi^2(\rho) \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma}{|\nabla_\gamma \rho|} \left(\frac{u}{\varphi(\rho)} \right) \right|^2 dz$$
$$= \int_{\mathbb{R}^n} \rho^{-(Q-2)} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma}{|\nabla_\gamma \rho|} \left(\rho^{\frac{Q-2-\alpha}{2}} u \right) \right|^2 dz.$$

Thus, after substituting these concrete functions into our main identity, we obtain the following power-weighted Grushin–Hardy identity:

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho^{-\alpha} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \left(\frac{Q-2-\alpha}{2} \right)^2 \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 \rho^{-2-\alpha} |u|^2 \, dz \\ &= \int_{\mathbb{R}^n} \rho^{-(Q-2)} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma}{|\nabla_\gamma \rho|} \left(\rho^{\frac{Q-2-\alpha}{2}} u \right) \right|^2 dz. \end{aligned} \tag{2.2.1}$$

This identity shows that the main result yields a concrete weighted Hardy-type identity when the power weight $V(\rho) = \rho^{-\alpha}$.

Special case 2.2.2 (Classical Hardy-type identity in the Grushin setting)**.** In this special case, we choose the weight function in the simplest form

$$V(\rho) = 1.$$

We also take

$$u \in C_0^\infty(B_R \backslash \{0\})$$

and choose the positive function

$$\varphi(\rho) = \rho^{-\frac{Q-2}{2}}.$$

From formula (2.1.2), we have

$$\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)' + \rho^{Q-1}W(\rho)\varphi(\rho) = 0.$$

Hence,

$$\rho^{Q-1}W(\rho)\varphi(\rho) = -\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'.$$

Therefore,

$$W(\rho) = -\frac{\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'}{\rho^{Q-1}\varphi(\rho)}.$$

Now we compute $W(\rho)$ for the chosen functions V and φ. Since

$$\varphi(\rho) = \rho^{-\frac{Q-2}{2}},$$

by the power rule for differentiation, we get

$$\varphi'(\rho) = -\frac{Q-2}{2}\rho^{-\frac{Q-2}{2}-1}.$$

Since $V(\rho) = 1$, we calculate

$$\begin{aligned}
\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)' &= \left[\rho^{Q-1}\left(-\frac{Q-2}{2}\rho^{-\frac{Q-2}{2}-1}\right)\right]' \\
&= \left[-\frac{Q-2}{2}\rho^{Q-2-\frac{Q-2}{2}}\right]' \\
&= \left[-\frac{Q-2}{2}\rho^{\frac{Q-2}{2}}\right]' \\
&= -\left(\frac{Q-2}{2}\right)^2\rho^{\frac{Q-2}{2}-1}.
\end{aligned}$$

Substituting this expression into the formula for $W(\rho)$, we obtain

$$\begin{aligned}
W(\rho) &= -\frac{-\left(\frac{Q-2}{2}\right)^2\rho^{\frac{Q-2}{2}-1}}{\rho^{Q-1}\rho^{-\frac{Q-2}{2}}} \\
&= \left(\frac{Q-2}{2}\right)^2\rho^{\frac{Q-2}{2}-1-\left(Q-1-\frac{Q-2}{2}\right)} \\
&= \left(\frac{Q-2}{2}\right)^2\rho^{\frac{Q-2}{2}-1-Q+1+\frac{Q-2}{2}} \\
&= \left(\frac{Q-2}{2}\right)^2\rho^{Q-2-Q} \\
&= \left(\frac{Q-2}{2}\right)^2\rho^{-2}.
\end{aligned}$$

Thus, the corresponding potential is

$$W(\rho) = \left(\frac{Q-2}{2}\right)^2 \rho^{-2}.$$

Now we substitute

$$V(\rho) = 1, \qquad \varphi(\rho) = \rho^{-\frac{Q-2}{2}}, \qquad W(\rho) = \left(\frac{Q-2}{2}\right)^2 \rho^{-2}$$

into our main identity.

The left-hand side becomes

$$\begin{aligned} LHS &= \int_{\mathbb{R}^n} V(\rho) \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 W(\rho) |u|^2 \, dz \\ &= \int_{\mathbb{R}^n} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 \rho^{-2} |u|^2 \, dz. \end{aligned}$$

Equivalently, since

$$\rho^{-2}|u|^2 = \frac{|u|^2}{\rho^2},$$

the left-hand side can be written as

$$LHS = \int_{\mathbb{R}^n} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|} \right|^2 dz - \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^n} |\nabla_\gamma \rho|^2 \frac{|u|^2}{\rho^2} \, dz.$$

Next, we calculate the weight on the right-hand side. By the choice of φ, we have

$$\varphi^2(\rho) = \left(\rho^{-\frac{Q-2}{2}}\right)^2 = \rho^{-(Q-2)}.$$

Since $V(\rho) = 1$, we obtain

$$V(\rho)\varphi^2(\rho) = 1 \cdot \rho^{-(Q-2)} = \rho^{-(Q-2)}.$$

Also,

$$\frac{u}{\varphi(\rho)} = \frac{u}{\rho^{-\frac{Q-2}{2}}} = \rho^{\frac{Q-2}{2}} u.$$

Hence, the right-hand side becomes

$$\begin{aligned} RHS &= \int_{\mathbb{R}^n} V(\rho)\varphi^2(\rho) \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma}{|\nabla_\gamma \rho|} \left(\frac{u}{\varphi(\rho)} \right) \right|^2 dz \\ &= \int_{\mathbb{R}^n} \rho^{-(Q-2)} \left| \frac{\nabla_\gamma \rho \cdot \nabla_\gamma}{|\nabla_\gamma \rho|} \left(\rho^{\frac{Q-2}{2}} u \right) \right|^2 dz. \end{aligned}$$

Thus, this special case gives the following classical Hardy-type identity in the

Grushin setting:

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}\right|^2dz-\left(\frac{Q-2}{2}\right)^2\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2\frac{|u|^2}{\rho^2}\,dz\\&=\int_{\mathbb{R}^n}\rho^{-(Q-2)}\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\rho^{\frac{Q-2}{2}}u\right)\right|^2dz.\end{aligned}\tag{2.2.2}$$

This identity shows that our main result contains the classical Hardy-type identity as a special case in the Grushin-type structure. In this case, the weight function is $V(\rho)=1$, and the potential term has the classical inverse-square form

$$W(\rho)=\left(\frac{Q-2}{2}\right)^2\rho^{-2}.$$

Special case 2.2.3 (Logarithmic improvement of the weighted Hardy identity)**.** Assume that $0<\rho<R$. In this special case, we choose

$$V(\rho)=\rho^{-\alpha}$$

and

$$\varphi(\rho)=\rho^{-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}.$$

From formula (2.1.2), we have

$$\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'+\rho^{Q-1}W(\rho)\varphi(\rho)=0.$$

Hence,

$$\rho^{Q-1}W(\rho)\varphi(\rho)=-\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'.$$

Therefore,

$$W(\rho)=-\frac{\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'}{\rho^{Q-1}\varphi(\rho)}.$$

Now we compute $W(\rho)$ for the chosen functions V and φ. Since

$$\varphi(\rho)=\rho^{-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}},$$

we first calculate $\varphi'(\rho)$. By the product rule,

$$\begin{aligned}
\varphi'(\rho) &= \left(\rho^{-\frac{Q-2-\alpha}{2}}\right)' \left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} + \rho^{-\frac{Q-2-\alpha}{2}} \left(\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}\right)' \\
&= -\frac{Q-2-\alpha}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} + \rho^{-\frac{Q-2-\alpha}{2}}\frac{1}{2}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}\left(\log\frac{R}{\rho}\right)' \\
&= -\frac{Q-2-\alpha}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} + \rho^{-\frac{Q-2-\alpha}{2}}\frac{1}{2}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}\left(-\frac{1}{\rho}\right) \\
&= -\frac{Q-2-\alpha}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} + \rho^{-\frac{Q-2-\alpha}{2}}\left(-\frac{1}{2\rho}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}\right)
\end{aligned}$$

Therefore,

$$\varphi'(\rho) = -\frac{Q-2-\alpha}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} - \frac{1}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}.$$

Now we calculate this part:

$$\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)'.$$

Since $V(\rho) = \rho^{-\alpha}$, we have

$$\rho^{Q-1}V(\rho) = \rho^{Q-1-\alpha}.$$

Thus,

$$\begin{aligned}
\rho^{Q-1}V(\rho)\varphi'(\rho) = \rho^{Q-1-\alpha}\Bigg(&-\frac{Q-2-\alpha}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} \\
&-\frac{1}{2}\rho^{-\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}\Bigg).
\end{aligned}$$

Combining the powers of ρ, we obtain

$$\begin{aligned}
\rho^{Q-1}V(\rho)\varphi'(\rho) = &-\frac{Q-2-\alpha}{2}\rho^{Q-\alpha-2-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} \\
&-\frac{1}{2}\rho^{Q-\alpha-2-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}.
\end{aligned}$$

Since

$$Q-\alpha-2-\frac{Q-2-\alpha}{2} = \frac{Q-2-\alpha}{2},$$

we get

$$\rho^{Q-1}V(\rho)\varphi'(\rho) = -\frac{Q-2-\alpha}{2}\rho^{\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} - \frac{1}{2}\rho^{\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}.$$

Now we differentiate this expression. First,

$$\begin{aligned}
&\left(-\frac{Q-2-\alpha}{2}\rho^{\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}\right)' \\
&= -\frac{Q-2-\alpha}{2}\left(\frac{Q-2-\alpha}{2}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} - \frac{1}{2}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}\right) \\
&= -\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} + \frac{Q-2-\alpha}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}.
\end{aligned}$$

Second,

$$\begin{aligned}
&\left(-\frac{1}{2}\rho^{\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}\right)' \\
&= -\frac{1}{2}\left[\frac{Q-2-\alpha}{2}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}} + \frac{1}{2}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{3}{2}}\right] \\
&= -\frac{Q-2-\alpha}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}} - \frac{1}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{3}{2}}.
\end{aligned}$$

Adding these two derivatives, the middle terms cancel:

$$\frac{Q-2-\alpha}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}} - \frac{Q-2-\alpha}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}} = 0.$$

Hence,

$$\left(\rho^{Q-1}V(\rho)\varphi'(\rho)\right)' = -\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} - \frac{1}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{3}{2}}.$$

Substituting this expression into the formula for $W(\rho)$, we obtain

$$W(\rho) = -\frac{-\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}} - \frac{1}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{3}{2}}}{\rho^{Q-1}\rho^{-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}.$$

Therefore,

$$W(\rho)=\frac{\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}{\rho^{Q-1-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}+\frac{\frac{1}{4}\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{3}{2}}}{\rho^{Q-1-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}.$$

For the first term, we get

$$\frac{\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}{\rho^{Q-1-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}=\rho^{\frac{Q-2-\alpha}{2}-1-Q+1+\frac{Q-2-\alpha}{2}}=\rho^{Q-2-\alpha-Q}=\rho^{-2-\alpha}.$$

For the second term, we get

$$\begin{aligned}\frac{\rho^{\frac{Q-2-\alpha}{2}-1}\left(\log\frac{R}{\rho}\right)^{-\frac{3}{2}}}{\rho^{Q-1-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}&=\rho^{\frac{Q-2-\alpha}{2}-1-Q+1+\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{-2}\\&=\rho^{-2-\alpha}\left(\log\frac{R}{\rho}\right)^{-2}.\end{aligned}$$

Thus,

$$W(\rho)=\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{-2-\alpha}+\frac{1}{4}\rho^{-2-\alpha}\left(\log\frac{R}{\rho}\right)^{-2}.$$

Now we substitute the functions

$$V(\rho)=\rho^{-\alpha},$$

$$\varphi(\rho)=\rho^{-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}},$$

and

$$W(\rho)=\left(\frac{Q-2-\alpha}{2}\right)^2\rho^{-2-\alpha}+\frac{1}{4}\rho^{-2-\alpha}\left(\log\frac{R}{\rho}\right)^{-2}$$

into our main identity.

The left-hand side becomes

$$\begin{aligned}LHS=&\int_{\mathbb{R}^n}\rho^{-\alpha}\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}\right|^2dz\\&-\left(\frac{Q-2-\alpha}{2}\right)^2\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2\rho^{-2-\alpha}|u|^2\,dz\\&-\frac{1}{4}\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2\rho^{-2-\alpha}\left(\log\frac{R}{\rho}\right)^{-2}|u|^2\,dz.\end{aligned}$$

Next, we calculate the weight on the right-hand side. By the choice of φ,

$$\begin{aligned}\varphi^2(\rho) &= \left(\rho^{-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}\right)^2\\ &= \rho^{-(Q-2-\alpha)}\log\frac{R}{\rho}.\end{aligned}$$

Therefore,

$$\begin{aligned}V(\rho)\varphi^2(\rho) &= \rho^{-\alpha}\rho^{-(Q-2-\alpha)}\log\frac{R}{\rho}\\ &= \rho^{-(Q-2)}\log\frac{R}{\rho}.\end{aligned}$$

Also,

$$\begin{aligned}\frac{u}{\varphi(\rho)} &= \frac{u}{\rho^{-\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{\frac{1}{2}}}\\ &= \rho^{\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}u.\end{aligned}$$

Hence, the right-hand side becomes

$$\begin{aligned}RHS &= \int_{\mathbb{R}^n} V(\rho)\varphi^2(\rho)\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\frac{u}{\varphi(\rho)}\right)\right|^2 dz\\ &= \int_{\mathbb{R}^n}\rho^{-(Q-2)}\log\frac{R}{\rho}\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\rho^{\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}u\right)\right|^2 dz.\end{aligned}$$

Thus, this special case gives the following logarithmically improved weighted Hardy identity:

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho^{-\alpha}\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma u}{|\nabla_\gamma\rho|}\right|^2 dz\\ &-\left(\frac{Q-2-\alpha}{2}\right)^2\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2\rho^{-2-\alpha}|u|^2\,dz\\ &\underbrace{-\frac{1}{4}\int_{\mathbb{R}^n}|\nabla_\gamma\rho|^2\rho^{-2-\alpha}\left(\log\frac{R}{\rho}\right)^{-2}|u|^2\,dz}_{\text{additional logarithmic improvement term}}\\ &=\int_{\mathbb{R}^n}\rho^{-(Q-2)}\log\frac{R}{\rho}\left|\frac{\nabla_\gamma\rho\cdot\nabla_\gamma}{|\nabla_\gamma\rho|}\left(\rho^{\frac{Q-2-\alpha}{2}}\left(\log\frac{R}{\rho}\right)^{-\frac{1}{2}}u\right)\right|^2 dz.\end{aligned}\tag{2.2.3}$$

Therefore, our main result not only gives the standard weighted Hardy identity, but also produces its logarithmic improvement in the Grushin-type setting.

In the case

$$\nabla_\gamma = \nabla, \qquad Q = n, \qquad \rho = |z|, \qquad |\nabla_\gamma \rho| = 1,$$

the special cases considered above reduce to the corresponding weighted Hardy-type identities in the classical Euclidean radial setting. Therefore, these examples are consistent with the Bessel pair approach studied by N. T. Duy, N. Lam, and N. Triet in [12].

Chapter 3

Conclusions and future work

3.1 Conclusions

This dissertation is devoted to the study of weighted Hardy-type inequalities and identities associated with the Baouendi–Grushin operator. The main result of the dissertation is the establishment of a new improved weighted Hardy identity in the Grushin-type setting by using the factorization method together with the theory of Bessel pairs.

In the obtained result, the radial derivative was represented in the form

$$\frac{\nabla_\gamma \rho \cdot \nabla_\gamma u}{|\nabla_\gamma \rho|},$$

which made it possible to derive an exact Hardy-type identity with an explicit nonnegative remainder term. Unlike the classical Euclidean case, the Baouendi–Grushin operator has a degenerate and anisotropic structure. Therefore, the corresponding functional inequalities require a substantially different analytical approach.

The proof of the main theorem was based on the construction of a suitable first-order differential operator and the computation of its formal adjoint. By expanding the composition T^*T and using the Bessel pair condition, we obtained not only a Hardy-type inequality, but a precise structural identity. This approach provides additional information about the geometry and analytical structure of the Grushin operator.

The obtained theorem generalizes the cylindrical weighted Hardy identities established by Nguyen Tuan Duy and Le Long Phi. In particular, the Euclidean radial variable and the standard gradient were replaced by the homogeneous quasi-norm ρ and the Grushin gradient ∇_γ. Consequently, the theorem extends the classical weighted Hardy identities from the Euclidean cylindrical setting to a more general degenerate Grushin-type framework.

Several important applications and special cases were derived from the main

theorem. By choosing appropriate weight functions V, W, and auxiliary functions φ, we obtained:

- power-weighted Hardy identities in the Grushin setting;
- the classical Hardy identity as a particular case;
- logarithmically improved Hardy identities with additional remainder terms.

These applications demonstrate that the main theorem provides a unified framework for deriving different classes of weighted Hardy-type inequalities and identities associated with degenerate operators.

An important advantage of the obtained identities is the presence of explicit remainder terms. Such remainder terms play a fundamental role in the study of sharp constants, stability estimates, and qualitative properties of solutions to partial differential equations involving singular potentials.

The results obtained in this dissertation open several possible directions for future research.

One natural continuation is the study of weighted Hardy inequalities in half-spaces and bounded domains related to the Baouendi–Grushin operator. In such domains, additional boundary terms may appear, leading to new improved inequalities and trace-type estimates. In particular, it would be interesting to investigate how the geometry of the boundary affects the optimal Hardy constants and remainder terms.

Another promising direction is the study of Hardy inequalities on domains with singular boundaries and anisotropic geometric structures adapted to the Grushin framework. Since the degeneracy of the operator changes the underlying metric geometry, many analytical phenomena may differ significantly from the Euclidean case.

It is also natural to investigate higher-order inequalities, especially Rellich-type inequalities associated with the Grushin operator. Since the present work is based on first-order factorization methods, similar techniques may potentially be extended to second-order and higher-order degenerate operators.

Another important topic is the study of fractional Hardy inequalities related to fractional Baouendi–Grushin operators and nonlocal degenerate operators. Such inequalities are closely connected with modern developments in fractional analysis and nonlocal partial differential equations.

Further possible research directions include:

- weighted Hardy inequalities with multiple singularities;
- critical Hardy inequalities and the study of optimal constants;
- uncertainty principle inequalities in Grushin-type spaces;
- weighted Sobolev and Hardy–Sobolev inequalities;

- Hardy inequalities on Carnot groups and subelliptic manifolds;
- nonlinear p-Hardy inequalities related to the Grushin p-Laplacian;
- spectral estimates for degenerate Schrödinger-type operators;
- existence and qualitative properties of extremal functions;
- applications to degenerate elliptic and parabolic equations.

Moreover, the methods developed in this dissertation may be applied to the study of qualitative properties of solutions to degenerate partial differential equations, especially equations involving singular potentials and anisotropic structures. Hardy-type inequalities often provide important tools for establishing a priori estimates, regularity properties, nonexistence results, and decay estimates for solutions.

From a broader perspective, this dissertation contributes to the theory of weighted functional inequalities in non-Euclidean and anisotropic geometries. The obtained results enrich the analysis related to degenerate differential operators and may serve as a basis for further investigations in harmonic analysis, subelliptic analysis, and geometric partial differential equations.

Thus, the results obtained in this dissertation not only extend the existing theory of weighted Hardy-type inequalities, but also provide a strong analytical foundation for future research related to degenerate operators, anisotropic structures, weighted inequalities, and non-Euclidean geometric analysis.

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