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S_n-MODULE STRUCTURES OF FREE ANTI-COMMUTATIVE ALGEBRA

THESIS

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Abdibek Ydyrys

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Dedication

This thesis is dedicated to:
My parents, SDU and many other for their support, help, sense of humour and useful comments for improving this project.

Abstract

An algebra with identity

$$ab = -ba$$

is called anti-commutative algebra. In this work we study S_n-module structures of free anti-commutative algebra of degree n.

Аңдатпа

$$ab = -ba$$

теңдігін қанағаттандыратын алгебра анти-коммутативті алгебра деп аталады. Бұл жұмыста біз дәрежесі n анти-коммутативті алгебраның симметриялық құрылымдарын зерттейміз.

Аннотация

Алгебра с тождеством

$$ab = -ba$$

называется анти-коммутативной алгеброй. В данной работе мы изучаем симметричные модульные структуры свободной анти-коммутативной алгебры степени n.

Contents

Introduction

1.1 Background and Significance

An algebra A over a field K is termed *anti-commutative* if for all elements $a, b \in A$, the identity $ab = -ba$ holds:

$$ab = -ba. \tag{1}$$

This anti-commutative property is a defining feature that distinguishes these algebras from other structures such as associative or commutative algebras. Anti-commutative algebras are significant in various mathematical and physical contexts. In mathematics, they appear naturally in the study of Lie algebras, which are essential for understanding symmetries and conservation laws. In theoretical physics, the algebra of fermionic operators in quantum mechanics is anti-commutative, reflecting the Pauli exclusion principle.

1.2 Literature Review

The exploration of anti-commutative algebras has a rich history. Early 20th-century foundational work by Sophus Lie and Wilhelm Killing laid the groundwork for understanding the structure and representation of these algebras. Further developments by Elie Cartan and Hermann Weyl established their importance in both mathematics and physics.

Among seminal papers, the research by Kass and Witthoft in 1970 stands out for its pioneering approach to anti-commutative algebras of degree 4, utilizing irreducible identities [4]. Their work, notable for applying Osborn's method, provided deeper insights into these algebras [5].

In the late 20th century, Yu. A. Bakhturin and C. Reutenauer made significant contributions to the understanding of free anti-commutative algebras and their module structures [2, 6]. Bakhturin's work on identical relations in Lie algebras and Reutenauer's research on free Lie algebras have been particularly influential.

Recently, Bremner's investigation into the S_n-module structures of free anti-commutative algebras for degrees $n \leq 7$ has provided comprehensive insights into the symmetries and decompositions of these algebras [1]. This study aims to extend these results to the case of degree 8.

1.3 State of the Art

Currently, the most up-to-date results in the field focus on the S_n-module structures for degrees $n \leq 7$. Bremner's comprehensive work provides a detailed analysis of these structures, paving the way for further exploration into higher degrees. Our research seeks to extend this body of knowledge to degree 8, utilizing advanced techniques from the theory of symmetric functions.

1.4 Research Gap

Despite significant advancements, the S_8-module structures of free anti-commutative algebras of degree 8 have not been thoroughly investigated. This gap represents a crucial area for further research, as understanding the S_8-module structures can provide deeper insights into the representation theory of symmetric groups and their applications in various mathematical and physical contexts.

1.5 Problem Statement, Research Questions, and Objectives

1.5.1 Problem Statement

The primary problem addressed in this research is the lack of comprehensive understanding of the S_8-module structures of free anti-commutative algebras of degree 8.

1.5.2 Research Questions

(i) What are the defining properties of free anti-commutative algebras of degree 8?

(ii) How does the symmetric group S_8 act on these algebras?

(iii) How can the resulting S_8-modules be decomposed into irreducible components?

(iv) What are the broader implications of these findings for representation theory and related fields?

1.5.3 Research Objectives

The primary objective of this thesis is to investigate the S_8-module structures of free anti-commutative algebras of degree 8. Specifically, we aim to:

(i) Define and analyze the concept of free anti-commutative algebras and their properties.

(ii) Examine the action of the symmetric group S_8 on these algebras and understand how this action induces a module structure.

(iii) Decompose the resulting S_8-modules into irreducible components using techniques from the theory of symmetric functions, such as the plethysm of Schur functions.

(iv) Explore the implications of our findings for the broader field of representation theory and related areas.

1.6 Results, Novelty, and Contributions

Our research has led to several significant findings:

(i) Detailed characterization of the S_8-module structures of free anti-commutative algebras of degree 8.

(ii) New decomposition techniques for these modules into irreducible components.

(iii) Enhanced understanding of the interplay between algebraic structures and symmetric group actions.

(iv) Implications for representation theory, combinatorics, algebraic topology, and mathematical physics.

1.7 Outline of Subsequent Chapters

This thesis is organized as follows:

- **Chapter 2: Main Results** - Presents the primary findings of the research, including detailed analyses and mathematical proofs.
- **Chapter 3: Conclusion** - Summarizes the research outcomes, discusses their implications, and suggests directions for future research.
- **Chapter 4: References** - Lists all the references cited in the thesis, providing a comprehensive bibliography.

Definition 1.1. A symmetric function in variables $x_1, x_2, \ldots, x_n$ is a function $f(x_1, x_2, \ldots, x_n)$ such that for any permutation σ of the indices $1, 2, \ldots, n$, we have:

$$f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

Example 1.2.

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3 \quad \text{and} \quad g(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$$

are both symmetric functions.

Definition 1.3. Schur functions are a special class of symmetric functions associated with a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and can be defined using the Jacobi-

Trudi identity:

$$s_\lambda(x_1, x_2, \ldots, x_n) = \det(h_{\lambda_i - i + j}(x_1, x_2, \ldots, x_n))_{1 \le i,j \le k}$$

where h_i are the complete homogeneous symmetric functions.

Example 1.4. For the partition $\lambda = (2, 1)$:

$$s_{(2,1)}(x_1, x_2) = \det \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = h_2 h_1 - h_3 h_0$$

Definition 1.5. The power sum symmetric functions p_k are defined as:

$$p_k = x_1^k + x_2^k + \cdots + x_n^k.$$

The power sum symmetric functions form an important basis for the ring of symmetric functions.

Example 1.6. For $k = 2$:

$$p_2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Definition 1.7. A free anti-commutative algebra over a field K is an algebra generated by elements $\{e_i\}$ subject to the relations:

$$e_i e_j = -e_j e_i \quad \text{for all} \quad i, j$$

and

$$e_i e_i = 0.$$

Example 1.8. The exterior algebra (or Grassmann algebra) generated by elements $\{e_1, e_2, \ldots, e_n\}$ with the wedge product $\wedge$ is a free anti-commutative algebra:

$$e_i \wedge e_j = -e_j \wedge e_i \quad \text{and} \quad e_i \wedge e_i = 0.$$

Definition 1.9. The symmetric group S_n acts on the free anti-commutative algebra generated by elements $\{e_1, e_2, \ldots, e_n\}$ by permuting the indices of the generators. The resulting algebra carries the structure of an S_n-module.

Example 1.10. For the exterior algebra generated by $\{e_1, e_2, e_3\}$, the action of the permutation $\sigma = (12) \in S_3$ on $e_1 \wedge e_2$ is given by:

$$\sigma \cdot (e_1 \wedge e_2) = e_2 \wedge e_1 = -e_1 \wedge e_2.$$

Definition 1.11. Given a subgroup H of a group G and a representation $\rho : H \to \mathrm{GL}(V)$, the induced representation $\mathrm{Ind}_H^G(\rho)$ is a representation of G constructed as follows. Consider the vector space:

$$\mathrm{Ind}_H^G(V) = \{f : G \to V \mid f(hg) = \rho(h)f(g) \text{ for all } h \in H, g \in G\}.$$

Example 1.12. Let $H = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ and $G = \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$, with ρ the trivial representation of H. Then $\mathrm{Ind}_H^G(\rho)$ is a representation of G on a space of dimension 2.

Definition 1.13. Plethysm is an operation on symmetric functions defined by the composition of symmetric functions. For symmetric functions f and g, the plethysm $f \circ g$ is defined by substituting the power sums of g into the power sums of f:

$$(f \circ g)(x_1, x_2, \ldots, x_n) = f(g(x_1), g(x_2), \ldots, g(x_n)).$$

Example 1.14. If $f = p_2$ and $g = e_2$, then the plethysm $p_2 \circ e_2$ is:

$$(p_2 \circ e_2)(x_1, x_2) = p_2(e_2(x_1, x_2)) = p_2(x_1x_2) = (x_1x_2)^2.$$

Definition 1.15. Let A be a set of vectors $\{a_1, \ldots, a_n\} \subseteq A$ and B be a set of vectors $\{b_1, \ldots, b_m\} \subseteq B$, where $m < n$. And if the following conditions are hold

(i) $\forall a_i = \sum_{j=1}^{m} \lambda_j b_j$

(ii) $\sum_{j=1}^{m} \lambda_j b_j = 0$ has only one unique solution and it is

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$$

Definition 1.16. Let A be a vector space over a field F, and let $v_1, \ldots, v_n \subseteq A$ be a set of vectors. We say that $v_1, \ldots, v_n$ is a basis for A if the following conditions hold:

(i) $v_1, \ldots, v_n$ is linearly independent, i.e., the only solution to the equation $\sum_{i=1}^{n} \lambda_i v_i = 0$ is the trivial solution $\lambda_1 = \cdots = \lambda_n = 0$.

(ii) $v_1, \ldots, v_n$ spans A, i.e., every vector in A can be written as a linear combination of the vectors in $v_1, \ldots, v_n$.

Definition 1.17. (Lexicographic order) Let A be an anti-commutative algebra and $x_1, \ldots, x_n \subseteq X$. We define an order on X as follows: $x_i > x_j$, if $i > j$. For $a, b \in A\langle X\rangle >$ and $a = a_1a_2, b = b_1b_2$, we say that a is greater than b and write $a > b$ if:

(i) if $deg(a) > deg(b)$;

(ii) if $deg(a) = deg(b), a_1 > b_1$;

(iii) if $deg(a) = deg(b), a_1 = b_1, a_2 > b_2$;

Remark. Let a be a monomial that is a basis element of $A\langle X\rangle$. We say that a is a basis element if $a_1 > a_2$ and both a_1 and a_2 are also basis monomials.

Example 1.18. For the anti-commutative algebra of $n = 2$, the basis monomial is only $[ab]$, since $[ba]$ is not a basis monomial according to the lexicographic order. It's important to note that by the property of anti-commutative algebra, $[ab]$ and $[ba]$ are the same.

Example 1.19. For the anti-commutative algebra of $n = 3$, the basis monomials are $[[ab]c]$, $[[ac]b]$, and $[[bc]a]$. If we consider other possible monomials, for instance, $[[ba]c] = -[[ab]c]$, $[[ca]b] = -[[ac]b]$, and $[[cb]a] = -[[bc]a]$, we can see that they are not basis monomials. This is because they do not satisfy (i) the lexicographic order and (ii) by the property of anti-commutative algebra $[ab] = -[ba]$, we can express these monomials using the basis elements.

Example 1.20. For the anti-commutative algebra of $n = 4$, we have 15 monomials of basis elements.

$$b_1 = [[[ab]c]d]$$
$$b_2 = [[[ab]d]c]$$
$$b_3 = [[[ac]b]d]$$
$$b_4 = [[[ac]d]b]$$
$$b_5 = [[[ad]b]c]$$
$$b_6 = [[[ad]c]b]$$
$$b_7 = [[[bc]a]d]$$
$$b_8 = [[[bc]d]a]$$
$$b_9 = [[[bd]a]c]$$
$$b_{10} = [[[bd]c]a]$$
$$b_{11} = [[[cd]a]b]$$
$$b_{12} = [[[cd]b]a]$$
$$b_{13} = [[ab][cd]]$$
$$b_{14} = [[ac][bd]]$$
$$b_{15} = [[ad][bc]]$$

We can verify that other monomials can be expressed in terms of these basis elements by the properties of anti-commutative algebra.

Example 1.21. The total number of basis monomials in the space P_5 is given by the binomial coefficient $\binom{6+5-1}{5-1} = \binom{10}{4} = 210$, since we have 6 variables (a, b, c, d, e, f) and we are choosing 4 of them to create a monomial. However, since the anti-commutative algebra means that the order of the variables matters, we have to divide by 2 for each monomial, since we could have also written it with the variables in the opposite order. Therefore, the actual number of basis monomials is $\frac{1}{2}\binom{10}{4} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = 105$. And monomials have form: $[[[**]*]*], [[**][**]]$, instead of asterisk will be a, b, c, d, e lexicographicly ordered.

Example 1.22. The number of basis monomials for P_6 is actually $\binom{6+6}{6} = 945$ and monomials have form: $[[[[**]*]*]*]$, $[[[**][**]]*]$, $[[[**]*][**]]$, instead of asterisk will be a, b, c, d, e lexicographicly ordered.

The table with numbers of basis elements

n	Number of basis elements
1	1
2	1
3	3
4	15
5	105
6	945

In the previous examples, we showed how we found basis monomials and how they look like.

Definition 1.23. Let S_n denote the symmetric group of permutations on n elements. A **characteristic map** $ch^n(\chi) : R^n \to \Lambda^n$ from the ring of symmetric functions to Schur functions is defined as follows:

For each permutation $\sigma \in S_n$, the characteristic map $\chi(\sigma)$ assigns a corresponding Schur function in Schur(n), denoted as $\chi(\sigma) \in \text{Schur}(n)$.

This mapping preserves certain properties or characteristics of the symmetric group elements within the context of Schur functions.

$$ch^n(\chi) = \sum z_\mu^{-1} \chi_\mu p_\mu$$

where, χ is character value, p is power sum of symmetric functions, z is conjugacy class number.

Example 1.24. Power sum of symmetric function for type $\alpha = (1,1,1), (2,1), (3)$

$$P_{(1,1,1)} = S_{(1,1,1)} + 2S_{(2,1)} + S_{(3)}$$

$$P_{(2,1)} = -S_{(1,1,1)} + S_{(3)}$$

$$P_{(3)} = S_{(1,1,1)} - S_{(2,1)} + S_{(3)}$$

$$A_3 \cong S_{(2,1)}$$

Because by calculation of S_n module for $n = 3$ generators of Free Anti-Commutative Algebra result is

$Ch^3(\chi_{\mathbb{A}}) = \frac{1}{2}(S_{(1,1,1)} - 2S_{(1,1,1)} + S_{(1,1,1)}) + \frac{1}{2}(2S_{(2,1)}) + 0(3S_{(3)}) = S_{(2,1)}$

Conjugacy Class	e (Identity)	(2-cycles)	(3-cycles)
Size	1	3	2
χ_1	1	1	1
χ_2	2	0	-1
χ_3	1	-1	1
$\chi_{\mathbb{A}}$	3	-1	0

Table 1.1: Character Table for S_3

Example 1.25. Power sum of symmetric function for type $\alpha = (1,1,1,1), (3,1), (2,2), (2,1), (1,1,1,1)$

$$P_{(4)} = S_{(4)} - S_{(3,1)} + S_{(2,1,1)} - S_{(1,1,1,1)}$$

$$P_{(3,1)} = S_{(4)} - S_{(2,2)} + S_{(1,1,1,1)}$$

$$P_{(2,2)} = S_{(4)} - S_{(3,1)} + S_{(2,2)} - S_{(2,1,1)} + S_{(1,1,1,1)}$$

$$P_{(2,1,1)} = S_{(4)} + S_{(3,1)} - S_{(2,1,1)} - S_{(1,1,1,1)}$$

$$P_{(1,1,1,1)} = S_{(4)} + 3S_{(3,1)} + 2S_{(2,2)} + 3S_{(2,1,1)} + S_{(1,1,1,1)}$$

Conjugacy Class	e (Identity)	(12)	(123)	(1234)	(12)(34)
Size	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2	3	1	0	-1	-1
χ_3	2	0	-1	0	2
χ_4	3	-1	0	1	-1
χ_5	1	-1	1	-1	1
$\chi_{[[[a,b]c]d]}$	12	-2	0	0	0
$\chi_{[[a,b][c,d]]}$	3	-1	-1	0	1

Table 1.2: Character Table for S_4

$$Ch^4(\chi) = S_{(3,1)} + S_{(2,1,1)}$$

With the same steps calculated for $n = 5, 6$

$$Ch^5 = S_{(4,1)} + S_{(3,2)} + S_{(3,1,1)} + S_{(2,2,1)} + S_{(2,1,1,1)}$$

$$Ch^6 = S_{(5,1)} + S_{(4,2)} + 2S_{(4,1,1)} + S_{(3,3)} + 3S_{(3,2,1)} + S_{(3,1,1,1)} + S_{(2,1,1,1,1)}$$

1.8 Symmetric Group Action on Algebras

The symmetric group S_n plays a significant role in the structure of free anti-commutative algebras. Let us explore how S_n acts on these algebras and how we can decompose the resulting modules.

Definition 1.26. The symmetric group S_n is the group of all permutations of the set $\{1, 2, \ldots, n\}$. A permutation $\sigma \in S_n$ can be represented as a bijective function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Theorem 1.27. *Let A be a free anti-commutative algebra generated by n elements. The action of the symmetric group S_n on A is defined by permuting the generators. This action induces a module structure on A over the group algebra $\mathbb{C}[S_n]$.*

Proof Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of generators for A. For each permutation $\sigma \in S_n$, define the action of σ on a generator x_i by $\sigma(x_i) = x_{\sigma(i)}$. Extend this action linearly to all elements of A. This defines a module structure on A over $\mathbb{C}[S_n]$. □

Example 1.28. Consider the free anti-commutative algebra generated by $n = 3$ elements x_1, x_2, x_3. The symmetric group S_3 has 6 elements: $(1), (12)$, $(13), (23), (123), (132)$. The action of S_3 on the generators is as follows:

$$\begin{aligned}
\sigma(x_1) &= x_{\sigma(1)}, \\
\sigma(x_2) &= x_{\sigma(2)}, \\
\sigma(x_3) &= x_{\sigma(3)},
\end{aligned}$$

for each $\sigma \in S_3$. For instance, if $\sigma = (12)$, then $\sigma(x_1) = x_2$, $\sigma(x_2) = x_1$, and $\sigma(x_3) = x_3$. Extend this action to the entire algebra by linearity and anti-commutativity.

1.9 Decomposition of S_n-Modules

To understand the structure of the free anti-commutative algebra as an S_n-module, we need to decompose it into irreducible components. This involves using tools from the representation theory of the symmetric group.

Definition 1.29. A module V over a group algebra $\mathbb{C}[G]$ is called irreducible if it has no proper non-zero submodules. That is, the only submodules of V are $\{0\}$ and V itself.

Theorem 1.30. *Every finite-dimensional module over the group algebra $\mathbb{C}[S_n]$ can be decomposed into a direct sum of irreducible modules.*

Proof This is a consequence of Maschke's theorem, which states that if G is a finite group and V is a finite-dimensional $\mathbb{C}[G]$-module, then V is completely reducible. That is, V can be written as a direct sum of irreducible submodules. □

Example 1.31. Consider the S_3-module structure of the free anti-commutative algebra generated by x_1, x_2, x_3. We can decompose this module into irreducible components by examining the action of S_3 on the basis elements. The irreducible representations of S_3 are well-known: the trivial representation, the sign representation, and the standard representation. By analyzing the action of S_3, we can determine how the module decomposes into these irreducibles.

In this chapter, we will explore specific examples and applications of the theoretical results developed in the previous chapters. We will consider concrete instances of free anti-commutative algebras and examine their S_n-module structures in detail.

1.10 Free Anti-Commutative Algebra of Degree 4

As discussed in the previous examples, the free anti-commutative algebra of degree 4 has 15 basis elements. We will now analyze the S_4-module structure of this algebra.

Example 1.32. Let A be the free anti-commutative algebra generated by x_1, x_2, x_3, x_4. The basis elements are given by:

$$\begin{aligned} b_1 &= [[[x_1x_2]x_3]x_4], \\ b_2 &= [[[x_1x_2]x_4]x_3], \\ &\vdots \\ b_{15} &= [[x_1x_4][x_2x_3]]. \end{aligned}$$

The symmetric group S_4 acts on these basis elements by permuting the indices. We can represent this action as a matrix and decompose the resulting representation into irreducible components.

1.11 Applications in Mathematical Physics

Free anti-commutative algebras and their S_n-module structures have applications in various areas of mathematical physics, including the study of supersymmetry and quantum field theory. The anti-commutative property is related to the behavior of fermionic fields, which obey the Pauli exclusion principle.

Example 1.33. In supersymmetric quantum mechanics, the algebra of fermionic creation and annihilation operators forms a free anti-commutative algebra. The S_n-module structure of this algebra can provide insights into the symmetries of the system and help classify the possible states.

Main Results

Let P_n^k be a space generated by kth binary tree with n leaves. Let S^λ be a Specht module for partition $\lambda \vdash n$. Let K be a filed of characteristic zero. All algebras, vector spaces and modules we consider will be over field K.

Theorem 2.1. *As S_8-module*

$$\begin{aligned} P_8^{(1)} \cong\ & S^{(1,1,1,1,1,1,1,1)} \oplus 6 \cdot S^{(2,1,1,1,1,1,1)} \oplus 15 \cdot S^{(2,2,1,1,1,1)} \oplus 19 \cdot S^{(2,2,2,1,1)} \\ & \oplus 15 \cdot S^{(3,1,1,1,1,1)} \oplus 40 \cdot S^{(3,2,1,1,1)} \oplus 40 \cdot S^{(3,2,2,1)} \oplus 30 \cdot S^{(3,3,1,1)} \oplus 21 \cdot S^{(3,3,2)} \\ & \oplus 20 \cdot S^{(4,1,1,1,1)} \oplus 45 \cdot S^{(4,2,1,1)} \oplus 26 \cdot S^{(4,2,2)} \oplus 30 \cdot S^{(4,3,1)} \oplus 5 \cdot S^{(4,4)} \\ & \oplus 15 \cdot S^{(5,1,1,1)} \oplus 24 \cdot S^{(5,2,1)} \oplus 9 \cdot S^{(5,3)} \oplus 6 \cdot S^{(6,1,1)} \oplus 5 \cdot S^{(6,2)} \oplus S^{(7,1)} \\ & \oplus 9 \cdot S^{(2,2,2,2)} \end{aligned}$$

$$\begin{aligned} P_8^{(2)} \cong\ & S^{(2,1,1,1,1,1,1)} \oplus 4 \cdot S^{(2,2,1,1,1,1)} \oplus 6 \cdot S^{(2,2,2,1,1)} \oplus 3 \cdot S^{(2,2,2,2)} \\ & \oplus 4 \cdot S^{(3,1,1,1,1,1)} \oplus 12 \cdot S^{(3,2,1,1,1)} \oplus 12 \cdot S^{(3,2,2,1)} \oplus 8 \cdot S^{(3,3,1,1)} \oplus 5 \cdot S^{(3,3,2)} \\ & \oplus 6 \cdot S^{(4,1,1,1,1)} \oplus 12 \cdot S^{(4,2,1,1)} \oplus 6 \cdot S^{(4,2,2)} \oplus 5 \cdot S^{(4,3,1)} \oplus 4 \cdot S^{(5,1,1,1)} \\ & \oplus 4 \cdot S^{(5,2,1)} \oplus S^{(6,1,1)} \end{aligned}$$

$$\begin{aligned} P_8^{(3)} \cong\ & S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)} \\ & \oplus 6 \cdot S^{(2,2,2,2)} \oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \\ & \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)} \oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \\ & \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)} \oplus 5 \cdot S^{(5,1,1,1)} \\ & \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)} \end{aligned}$$

$$P_8^{(4)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(5)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)}$$
$$\oplus 6 \cdot S^{(2,2,2,2)} \oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)}$$
$$\oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)} \oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)}$$
$$\oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)} \oplus 5 \cdot S^{(5,1,1,1)}$$
$$\oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(6)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,1,1,1,1,1,1)} \oplus 6 \cdot S^{(2,2,1,1,1,1)} \oplus 7 \cdot S^{(2,2,2,1,1)}$$
$$\oplus 4 \cdot S^{(3,1,1,1,1,1)} \oplus 12 \cdot S^{(3,2,1,1,1)} \oplus 11 \cdot S^{(3,2,2,1)} \oplus 9 \cdot S^{(3,3,1,1)} \oplus 5 \cdot S^{(3,3,2)}$$
$$\oplus 4 \cdot S^{(4,1,1,1,1)} \oplus 11 \cdot S^{(4,2,1,1)} \oplus 5 \cdot S^{(4,2,2)} \oplus 7 \cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus 3 \cdot S^{(5,1,1,1)} \oplus 4 \cdot S^{(5,2,1)} \oplus S^{(5,3)} \oplus S^{(6,1,1)} \oplus 3 \cdot S^{(2,2,2,2)}$$

$$P_8^{(7)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)}$$
$$\oplus 6 \cdot S^{(2,2,2,2)} \oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)}$$
$$\oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)} \oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)}$$
$$\oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)} \oplus 5 \cdot S^{(5,1,1,1)}$$
$$\oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(8)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(9)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 4 \cdot S^{(2,1,1,1,1,1,1)} \oplus 8 \cdot S^{(2,2,1,1,1,1)} \oplus 9 \cdot S^{(2,2,2,1,1)} \oplus 4 \cdot S^{(2,2,2,2)}$$
$$\oplus 6 \cdot S^{(3,1,1,1,1,1)} \oplus 14 \cdot S^{(3,2,1,1,1)} \oplus 13 \cdot S^{(3,2,2,1)} \oplus 9 \cdot S^{(3,3,1,1)} \oplus 6 \cdot S^{(3,3,2)}$$
$$\oplus 4 \cdot S^{(4,1,1,1,1)} \oplus 10 \cdot S^{(4,2,1,1)} \oplus 5 \cdot S^{(4,2,2)} \oplus 7 \cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus 4 \cdot S^{(5,1,1,1)} \oplus 5 \cdot S^{(5,2,1)} \oplus S^{(5,3)} \oplus S^{(6,1,1)}$$

$$P_8^{(10)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)}$$
$$\oplus 6 \cdot S^{(2,2,2,2)} \oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)}$$
$$\oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)} \oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)}$$
$$\oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)} \oplus 5 \cdot S^{(5,1,1,1)}$$
$$\oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(11)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(12)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)}$$
$$\oplus 6 \cdot S^{(2,2,2,2)} \oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)}$$
$$\oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)} \oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)}$$
$$\oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)} \oplus 5 \cdot S^{(5,1,1,1)}$$
$$\oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$P_8^{(13)} \cong S^{(2,1,1,1,1,1,1)} \oplus 3 \cdot S^{(2,2,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)} \oplus 2 \cdot S^{(2,2,2,2)}$$
$$\oplus 3 \cdot S^{(3,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 7 \cdot S^{(3,2,2,1)} \oplus 4 \cdot S^{(3,3,1,1)} \oplus 3 \cdot S^{(3,3,2)}$$
$$\oplus 3 \cdot S^{(4,1,1,1,1)} \oplus 5 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 2 \cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}$$

$$P_8^{(14)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 4 \cdot S^{(2,1,1,1,1,1,1)} \oplus 8 \cdot S^{(2,2,1,1,1,1)} \oplus 9 \cdot S^{(2,2,2,1,1)} \oplus 4 \cdot S^{(2,2,2,2)}$$
$$\oplus 6 \cdot S^{(3,1,1,1,1,1)} \oplus 14 \cdot S^{(3,2,1,1,1)} \oplus 13 \cdot S^{(3,2,2,1)} \oplus 9 \cdot S^{(3,3,1,1)} \oplus 6 \cdot S^{(3,3,2)}$$
$$\oplus 4 \cdot S^{(4,1,1,1,1)} \oplus 9 \cdot S^{(4,2,1,1)} \oplus 5 \cdot S^{(4,2,2)} \oplus 6 \cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus 4 \cdot S^{(5,1,1,1)} \oplus 5 \cdot S^{(5,2,1)} \oplus S^{(5,3)} \oplus S^{(6,1,1)}$$

$$P_8^{(15)} \cong S^{(2,1,1,1,1,1,1)} \oplus 2 \cdot S^{(2,2,1,1,1,1)} \oplus 3 \cdot S^{(2,2,2,1,1)} \oplus S^{(2,2,2,2)}$$
$$\oplus 2 \cdot S^{(3,1,1,1,1,1)} \oplus 4 \cdot S^{(3,2,1,1,1)} \oplus 4 \cdot S^{(3,2,2,1)} \oplus 2 \cdot S^{(3,3,1,1)} \oplus 2 \cdot S^{(3,3,2)}$$
$$\oplus S^{(4,1,1,1,1)} \oplus 2 \cdot S^{(4,2,1,1)} \oplus S^{(4,2,2)} \oplus S^{(4,3,1)}$$

$$P_8^{(16)} \cong S^{(1,1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,1)}$$
$$\oplus 10 \cdot S^{(3,1,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)}$$
$$\oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)}$$
$$\oplus 5 \cdot S^{(5,1,1,1)} \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$
$$\oplus 6 \cdot S^{(2,2,2,2)}$$

$$P_8^{(17)} \cong S^{(1,1,1,1,1,1,1,1,1)} \oplus 2 \cdot S^{(2,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,2,1,1,1,1,1)} \oplus 4 \cdot S^{(2,2,2,1,1)}$$
$$\oplus 2 \cdot S^{(3,1,1,1,1,1,1)} \oplus 7 \cdot S^{(3,2,1,1,1)} \oplus 6 \cdot S^{(3,2,2,1)} \oplus 6 \cdot S^{(3,3,1,1)} \oplus 2 \cdot S^{(3,3,2)}$$
$$\oplus 2 \cdot S_8^{(4,1,1,1,1)} \oplus 4 \cdot S^{(4,2,1,1)} \oplus 3 \cdot S^{(4,2,2)} \oplus 3 \cdot S^{(4,3,1)} \oplus S^{(4,4)}$$
$$\oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)} \oplus 3 \cdot S^{(2,2,2,2)}$$

$$P_8^{(18)} \cong S^{(1,1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)}$$
$$\oplus 6 \cdot S^{(2,2,2,2)} \oplus 10 \cdot S^{(3,1,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)}$$
$$\oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)} \oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)}$$
$$\oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)} \oplus 5 \cdot S^{(5,1,1,1)}$$
$$\oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

$$
\begin{aligned}
P_8^{(19)} \cong\ & S^{(2,1,1,1,1,1,1)} \oplus 3\cdot S^{(2,2,1,1,1,1)} \oplus 4\cdot S^{(2,2,2,1,1)} \oplus 2\cdot S^{(2,2,2,2)} \\
& \oplus 3\cdot S^{(3,1,1,1,1,1)} \oplus 7\cdot S^{(3,2,1,1,1)} \oplus 7\cdot S^{(3,2,2,1)} \oplus 4\cdot S^{(3,3,1,1)} \oplus 3\cdot S^{(3,3,2)} \\
& \oplus 3\cdot S^{(4,1,1,1,1)} \oplus 5\cdot S^{(4,2,1,1)} \oplus 3\cdot S^{(4,2,2)} \oplus 2\cdot S^{(4,3,1)} \oplus S_8^{(5,1,1,1)} \oplus S^{(5,2,1)}
\end{aligned}
$$

$$
\begin{aligned}
P_8^{(20)} \cong\ & S^{(1,1,1,1,1,1,1,1)} \oplus 5\cdot S^{(2,1,1,1,1,1,1)} \oplus 11\cdot S^{(2,2,1,1,1,1)} \oplus 13\cdot S^{(2,2,2,1,1)} \\
& \oplus 10\cdot S^{(3,1,1,1,1,1)} \oplus 24\cdot S^{(3,2,1,1,1)} \oplus 23\cdot S^{(3,2,2,1)} \oplus 16\cdot S^{(3,3,1,1)} \oplus 11\cdot S^{(3,3,2)} \\
& \oplus 10\cdot S^{(4,1,1,1,1)} \oplus 21\cdot S^{(4,2,1,1)} \oplus 12\cdot S^{(4,2,2)} \oplus 13\cdot S^{(4,3,1)} \oplus 2\cdot S^{(4,4)} \\
& \oplus 5\cdot S^{(5,1,1,1)} \oplus 8\cdot S^{(5,2,1)} \oplus 3\cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)} \oplus 6\cdot S^{(2,2,2,2)}
\end{aligned}
$$

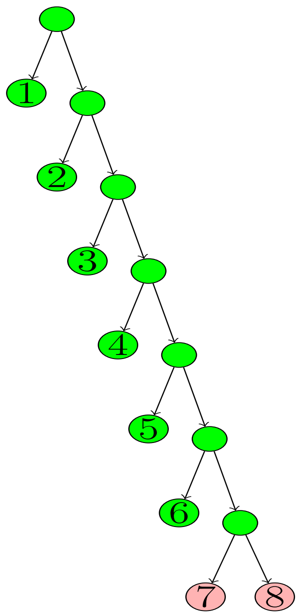
$$
\begin{aligned}
P_8^{(21)} \cong\ & S^{(2,1,1,1,1,1,1)} \oplus 3\cdot S^{(2,2,1,1,1,1)} \oplus 4\cdot S^{(2,2,2,1,1)} \oplus 2\cdot S^{(2,2,2,2)} \\
& \oplus 3\cdot S^{(3,1,1,1,1,1)} \oplus 7\cdot S^{(3,2,1,1,1)} \oplus 7\cdot S^{(3,2,2,1)} \oplus 4\cdot S^{(3,3,1,1)} \oplus 3\cdot S^{(3,3,2)} \\
& \oplus 3\cdot S^{(4,1,1,1,1)} \oplus 5\cdot S^{(4,2,1,1)} \oplus 3\cdot S^{(4,2,2)} \oplus 2\cdot S^{(4,3,1)} \oplus S^{(5,1,1,1)} \oplus S^{(5,2,1)}
\end{aligned}
$$

$$
P_8^{(22)} \cong S^{(2,2,2,1,1)} \oplus S^{(3,1,1,1,1,1)} \oplus S^{(3,2,1,1,1)} \oplus S^{(3,2,2,1)} \oplus S^{(3,3,2)} \oplus S^{(4,2,1,1)}.
$$

We will consider proof for Tree 1, 2 and 22.

First, let's label each leaf with numbers $\{1, 2, \ldots, 8\}$ from bottom to top. See Fig. 2.1

Figure 2.1: Binary Tree 1

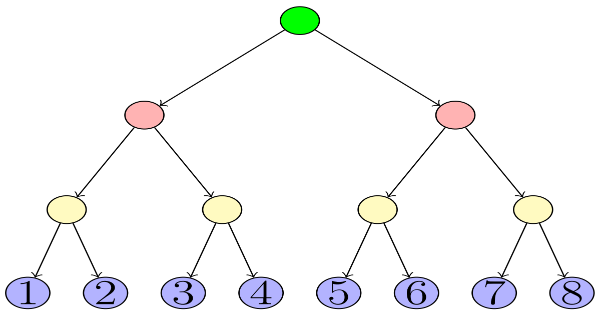


Obviously, the group of symmetries of the binary tree 1 is $Sym(T_1) = \langle (12) \rangle$. Moreover, the binary tree 1 corresponds to the following non-associative monomial

$$T_1 := (((((((12)3)4)5)6)7)8).$$

Now, we label each leaf of T_{22} by numbers $\{1, 2, \cdots, 8\}$ in the following way

Figure 2.2: Binary Tree 22



Obviously, the group of symmetries of binary tree T_{22} is

$$Sym(T_{22}) = \langle (12), (13)(24), (15)(26)(37)(48) \rangle =$$

$\{(1), (12), (13)(24), (1423), (15)(26)(37)(48), (1625)(37)(48), (1735)(2846),$
$(18462735), (56), (12)(56), (13)(24)(56), (1423)(56), (1526)(37)(48),$
$(16)(25)(37)(48), (17352846), (1846)(2735), (57)(68), (12)(57)(68),$
$(13)(24)(57)(68), (1423)(57)(68), (1537)(2648), (16482537),$
$(17)(28)(35)(46), (1827)(35)(46), (5867), (12)(5867), (13)(24)(5867),$
$(1423)(5867), (15482637), (1637)(2548), (17)(28)(3546), (1827)(3546),$
$(1324), (14)(23), (15)(26)(3847), (1625)(3847), (17362845),$
$(12)(34)(56), (1324)(56), (14)(23)(56), (1526)(3847), (16)(25)(3847),$
$(18452736), (34)(57)(68), (12)(34)(57)(68), (1324)(57)(68),$
$(1647)(2538), (17)(28)(3645), (1827)(3645), (34)(5867), (1728)(36)(45),$
$(14)(23)(5867), (1547)(2638), (16382547), (17)(28)(36)(45),$
$(12)(78), (13)(24)(78), (1423)(78), (15)(26)(3748), (1625)(3748),$
$(56)(78), (12)(56)(78), (13)(24)(56)(78), (1423)(56)(78),$
$(1746)(2835), (18352746), (5768), (12)(5768), (13)(24)(5768),$
$(1648)(2537), (1728)(35)(46), (18)(27)(35)(46), (58)(67),$
$(1423)(58)(67), (1548)(2637), (16372548), (1728)(3546), (18)(27)(3546),$
$(12)(34)(78), (1324)(78), (14)(23)(78), (15)(26)(38)(47),$
$(18362745), (34)(56)(78), (12)(34)(56)(78), (1324)(56)(78),$
$(16)(25)(38)(47), (17452836), (1836)(2745), (34)(5768),$
$(14)(23)(5768), (1538)(2647), (16472538), (1728)(3645),$
$(12)(34)(58)(67), (1324)(58)(67), (14)(23)(58)(67), (15472638),$
$(14)(23)(57)(68), (15382647), (12)(34)(5867), (1324)(5867),$
$(1827)(36)(45), (78), (17462835), (1835)(2746), (1526)(3748), (16)(25)(3748),$
$(1423)(5768), (15372648), (12)(58)(67), (13)(24)(58)(67), (34)(78),$
$(1625)(38)(47), (1745)(2836), (14)(23)(56)(78), (1526)(38)(47)$
$(12)(34)(5768), (1324)(5768), (18)(27)(3645), (34)(58)(67), (1638)(2547),$
$(18)(27)(36)(45), (34), (12)(34), (1845)(2736), (34)(56), (1736)(2845)\}$

Lemma 2.2. *Let T_1 be the binary tree 1 and let $\sigma \in S_2$ be a permutation. Then*

$$\sigma T_1 = T_1, \ \textit{if } \sigma = (1),$$

$$\sigma T_1 = -T_1, \ \textit{if } \sigma = (12).$$

Proof *If $\sigma = (1)$, then it is clear. If $\sigma = (12)$, then*

$$(12) \circ (((((((12)3)4)5)6)7)8) = (((((((21)3)4)5)6)7)8) = (1) =$$

$$-(((((((12)3)4)5)6)7)8).$$

□

Lemma 2.3. *Let S_2 be the symmetric group on set $\{1, 2\}$. Then*

$$Sym(T_{22}) \cong S_2 \wr (S_2 \wr S_2).$$

Proof Set

$$a = (12), b = (13)(24), c = (15)(26)(37)(48).$$

Define bijective function $f : Sym(T_{22}) \to S_2 \wr (S_2 \wr S_2)$ in the following way

$$f(a) = \{[(12), (1); (1)], [(1), (1); (1)]; (1)\},$$

$$f(b) = \{[(1), (1); (12)], [(1), (1); (1)]; (1)\},$$

$$f(c) = \{[(1), (1); (1)], [(1), (1); (1)]; (12)\}.$$

Now let us show that the function f is a group morphism.
Applying the function f to $a \circ b$, $b \circ a$, $a \circ c$, $c \circ a$, $b \circ c$ and $c \circ b$ we get

$$f(a \circ b) = f(1324) = \{[(1), (12); (12)], [(1), (1); (1)]; (1)\};$$

$$f(b \circ a) = f(1423) = \{[(1), (12); (12)], [(1), (1); (1)]; (1)\};$$

$$f(a \circ c) = f((1526)(37)(48)) = \{[(1), (1); (1)], [(12), (1); (1)]; (12)\};$$

$$f(c \circ a) = f((1625)(37)(48)) = \{[(12), (1); (1)], [(1), (1); (1)]; (12)\};$$

$$f(b \circ c) = f((1537)(2648)) = \{[(1), (1); (12)], [(1), (1); (1)]; (12)\};$$

$$f(c \circ b) = f((1735)(4628)) = \{[(1), (1); (1)], [(12), (12); (12)]; (12)\}.$$

Now, let's compute

$$\begin{aligned} f(a)\cdot f(b) &= \{[(12),(1);(1)],[(1),(1);(1)];(1)\}\cdot\{[(1),(1);(12)],[(1),(1);(1)];(1)\} \\ &= \{[(1),(12);(12)],[(1),(1);(1)];(1)\}\,. \end{aligned}$$

$$\begin{aligned} f(b)\cdot f(a) &= \{[(1),(1);(12)],[(1),(1);(1)];(1)\}\cdot\{[(12),(1);(1)],[(1),(1);(1)];(1)\} \\ &= \{[(1),(12);(12)],[(1),(1);(1)];(1)\} \end{aligned}$$

$$\begin{aligned} f(a)\cdot f(c) &= \{[(12),(1);(1)],[(1),(1);(1)];(1)\}\cdot\{[(1),(1);(1)],[(1),(1);(1)];(12)\} \\ &= \{[(1),(1);(1)],[(12),(1);(1)];(12)\} \end{aligned}$$

$$\begin{aligned} f(c)\cdot f(a) &= \{[(1),(1);(1)],[(1),(1);(12)];(1)\}\cdot\{[(12),(1);(1)],[(1),(1);(1)];(1)\} \\ &= \{[(12),(1);(1)],[(1),(1);(1)];(12)\} \end{aligned}$$

$$\begin{aligned} f(b)\cdot f(c) &= \{[(1),(1);(12)],[(1),(1);(1)];(1)\}\cdot\{[(1),(1);(1)],[(1),(1);(1)];(12)\} \\ &= \{[(1),(1);(12)],[(1),(1);(1)];(12)\} \end{aligned}$$

$$\begin{aligned} f(c)\cdot f(b) &= \{[(1),(1);(1)],[(1),(1);(1)];(12)\}\cdot\{[(1),(1);(12)],[(1),(1);(1)];(1)\} \\ &= \{[(1),(1);(1)],[(12),(12);(12)];(12)\} \end{aligned}$$

Hence,

$$f(a\circ b) = f(a)\cdot f(b);$$

$$f(b\circ a) = f(b)\cdot f(a);$$

$$f(a\circ c) = f(a)\cdot f(c);$$

$$f(c\circ a) = f(c)\cdot f(a);$$

$$f(b\circ c) = f(b)\cdot f(c);$$

$$f(c\circ b) = f(c)\cdot f(b).$$

□

Lemma 2.4. *Let T_{22} be the binary tree and let $\sigma \in S_2$ be a permutation. Then*

$$\sigma T_{22} = T_{22}, \text{ if } \sigma = (1);$$

$$\sigma T_{22} = -T_{22}, \text{ if } \sigma = (12);$$

$$\sigma T_{22} = -T_{22}, \text{ if } \sigma = (13)(24);$$

$$\sigma T_{22} = -T_{22}, \text{ if } \sigma = (15)(26)(37)(48).$$

Proof if $\sigma = (1)$, then it is obvious. If $\sigma = (12)$, then

$$(12)\circ((12)(34))((56)(78)) = ((21)(34))((56)(78)) = (1) = -((12)(34))((56)(78))$$

If $\sigma = (13)(24)$, then

$$(13)(24) \circ ((12)(34))((56)(78)) = ((34)(12))((56)(78)) = (1) =$$

$$= -((12)(34))((56)(78))$$

If $\sigma = (15)(26)(37)(48)$, then

$$(15)(26)(37)(48) \circ ((12)(34))((56)(78)) = ((56)(78))((12)(34)) = (1) =$$

$$= -((12)(34))((56)(78))$$

□

Proof of Theorem Using Lemma (2.2) we get

$$P_8^1 \cong Ind_{S_1\times S_1\times S_1\times S_1\times S_1\times S_1\times S_2}^{S_8}(\mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_2}^-) \cong$$

$$S^{(1,1,1,1,1,1,1,1)} \oplus 6 \cdot S^{(2,1,1,1,1,1,1)} \oplus 15 \cdot S^{(2,2,1,1,1,1)} \oplus 19 \cdot S^{(2,2,2,1,1)} \oplus 9 \cdot S^{(2,2,2,2)}$$
$$\oplus 15 \cdot S^{(3,1,1,1,1,1)} \oplus 40 \cdot S^{(3,2,1,1,1)} \oplus 40 \cdot S^{(3,2,2,1)} \oplus 30 \cdot S^{(3,3,1,1)} \oplus 21 \cdot S^{(3,3,2)}$$
$$\oplus 20 \cdot S^{(4,1,1,1,1)} \oplus 45 \cdot S^{(4,2,1,1)} \oplus 26 \cdot S^{(4,2,2)} \oplus 30 \cdot S^{(4,3,1)} \oplus 5 \cdot S^{(4,4)}$$
$$\oplus 15 \cdot S^{(5,1,1,1)} \oplus 24 \cdot S^{(5,2,1)} \oplus 9 \cdot S^{(5,3)} \oplus 6 \cdot S^{(6,1,1)} \oplus 5 \cdot S^{(6,2)} \oplus S^{(7,1)}.$$

Using Lemma (2.3) and Lemma (2.4) we get

$$P_8^{22} \cong Ind_{S_2 \wr (S_2 \wr S_2)}^{s_8} (\mathbf{1}_{S_2}^- \otimes (\mathbf{1}_{S_2}^- \otimes \mathbf{1}_{S_2}^-)).$$

To calculate this induced representation we need the theory of symmetric functions. We apply characteristic map on the induced representation above we get the following plethysm of Schur functions

$$Ch(Ind_{S_2 \wr (S_2 \wr S_2)}^{s_8} (\mathbf{1}_{S_2}^- \otimes (\mathbf{1}_{S_2}^- \otimes \mathbf{1}_{S_2}^-)) =$$

$$= s_{(1,1)} \circ (s_{(1,1)} \circ s_{(1,1)}).$$

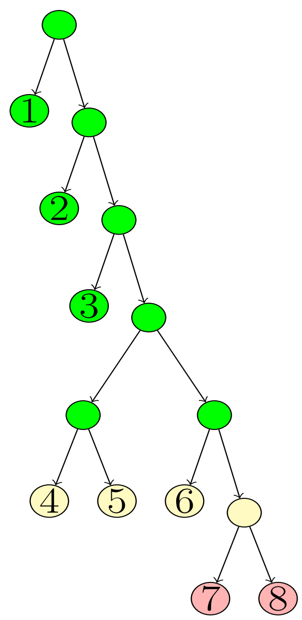
Calculating the plethysm above we get

$$s_{(1,1)} \circ (s_{(1,1)} \circ s_{(1,1)}) = s_{(2,2,2,1,1)} + s_{(3,1,1,1,1,1)} + s_{(3,2,2,1)} + s_{(4,2,1,1)}$$

□

Now let's consider binary tree $2(T_2)$. See Fig 2.3 Obviously, the group of sym-

Figure 2.3: Binary Tree 2



metries of the binary tree 2 is $Sym(T_2) = \langle (12), (13)(24) \rangle$ and non-associative monomial corresponds to tree 2

$$T_2 := (1(2(3((45)(6(78))))))$$

let's rewrite monomial in the form, since composition with generators of group of symmetries is more convenient

$$T_2 := (5(6(7((12)(8(34))))))$$

Lemma 2.5. *Let T_2 be the binary tree 2 and let $\sigma \in S_2$ be a permutation. Then*

$$\sigma T_2 = T_2, \ \textit{if } \sigma = (1);$$

$$\sigma T_2 = -T_2, \ \textit{if } \sigma = (12);$$

$$\sigma T_2 = -T_2, \ \textit{if } \sigma = (13)(24);$$

Proof if $\sigma = (1)$, then it is obvious. If $\sigma = (12)$, then

$$(12) \circ (5(6(7((12)(8(34)))))) = (5(6(7((21)(8(34)))))) = (1) =$$

$$= -(5(6(7((12)(8(34))))))$$

If $\sigma = (13)(24)$, then

$$(13)(24) \circ (5(6(7((12)(8(34)))))) = (5(6(7((34)(8(12)))))) = (1) =$$

$$= -(5(6(7((12)(8(34))))))$$

□

Lemma 2.6. *Let S_2 be the symmetric group on set $\{1, 2\}$. Then*

$$Sym(T_2) \cong S_1 \times S_1 \times S_1 \times S_1 \times S_2 \times S_2.$$

Proof of Theorem Using Lemma (2.5) we get

$$P_8^2 \cong Ind_{S_1 \times S_1 \times S_1 \times S_1 \times S_2 \times S_2}^{S_8}(\mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_1}^- \otimes \mathbf{1}_{S_2}^- \otimes \mathbf{1}_{S_2}^-) \cong$$

$$P_8^{(2)} \cong S^{(2,1,1,1,1,1,1)} \oplus 4 \cdot S^{(2,2,1,1,1,1)} \oplus 6 \cdot S^{(2,2,2,1,1)} \oplus 3 \cdot S^{(2,2,2,2)} \oplus 4 \cdot S^{(3,1,1,1,1,1)} \oplus 12 \cdot S^{(3,2,1,1,1)} \oplus 12 \cdot S^{(3,2,2,1)} \oplus 8 \cdot S^{(3,3,1,1)} \oplus 5 \cdot S^{(3,3,2)} \oplus 6 \cdot S^{(4,1,1,1,1)} \oplus 12 \cdot S^{(4,2,1,1)} \oplus 6 \cdot S^{(4,2,2)} \oplus 5 \cdot S^{(4,3,1)} \oplus 4 \cdot S^{(5,1,1,1)} \oplus 4 \cdot S^{(5,2,1)} \oplus S^{(6,1,1)}$$

Using Lemma (2.6) we get

$$P_8^2 \cong Ind^{s_8}_{S_1\times S_1\times S_1\times S_1\times S_2\times S_2}(\mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2} \otimes \mathbf{1}^-_{S_2}).$$

To calculate this induced representation we need the theory of symmetric functions. We apply characteristic map on the induced representation above we get the following Schur functions

$$Ch(Ind^{s_8}_{S_1\times S_1\times S_1\times S_1\times S_2\times S_2}(\mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2} \otimes \mathbf{1}^-_{S_2})) =$$

$$= s_{(1)} \times s_{(1)} \times s_{(1)} \times s_{(1)} \times s_{(1,1)} \times s_{(1,1)}.$$

□

The rest $P_8^3 - P_8^{21}$ are proven similar way. And binary trees 3, 5, 7, 10, 12, and 18 have the same structure, let's break it down with detailed explanations:

1. **Symmetric Group Correspondence**:

$$Sym(T_3) \cong S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$$

   This line signifies the isomorphism between the symmetric group $Sym(T_3)$ and the direct product of symmetric groups $S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$. It suggests that the group $Sym(T_3)$ is essentially composed of permutations that can be decomposed into permutations from each of these individual symmetric groups.

2. **Induced Representation**:

$$P_8^3 \cong Ind^{s_8}_{S_1\times S_1\times S_1\times S_2\times S_1\times S_2}(\mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2})$$

   This equation denotes the induced representation P_8^3 of degree 8 from the direct product of symmetric groups $S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$, denoted as s_8. It is induced by the tensor product of irreducible representations $\mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2}$.

3. **Character of Induced Representation**:

$$Ch(Ind^{s_8}_{S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2}(\mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2} \otimes \mathbf{1}^-_{S_1} \otimes \mathbf{1}^-_{S_2}) = \\ = s_{(1)} \times s_{(1)} \times s_{(1)} \times s_{(1,1)} \times s_{(1)} \times s_{(1,1)}$$

This part calculates the character of the induced representation described above. The character is a function that associates each group element with a complex number, capturing information about the representation.

4. **Decomposition of Induced Representation**:

$$P_8^{(3)} \cong S^{(1,1,1,1,1,1,1,1)} \oplus 5 \cdot S^{(2,1,1,1,1,1,1)} \oplus 11 \cdot S^{(2,2,1,1,1,1)} \oplus 13 \cdot S^{(2,2,2,1,10)} \\ \oplus 6 \cdot S^{(2,2,2,2)} \oplus 10 \cdot S^{(3,1,1,1,1,1)} \oplus 24 \cdot S^{(3,2,1,1,1)} \oplus 23 \cdot S^{(3,2,2,1)} \\ \oplus 16 \cdot S^{(3,3,1,1)} \oplus 11 \cdot S^{(3,3,2)} \oplus 10 \cdot S^{(4,1,1,1,1)} \oplus 21 \cdot S^{(4,2,1,1)} \\ \oplus 12 \cdot S^{(4,2,2)} \oplus 13 \cdot S^{(4,3,1)} \oplus 2 \cdot S^{(4,4)} \oplus 5 \cdot S^{(5,1,1,1)} \\ \oplus 8 \cdot S^{(5,2,1)} \oplus 3 \cdot S^{(5,3)} \oplus S^{(6,1,1)} \oplus S^{(6,2)}$$

Here, the result presents the decomposition of the induced representation $P_8^{(3)}$ into irreducible representations. Each term in the decomposition corresponds to a different irreducible representation, with coefficients indicating how many times each representation appears in the decomposition. For example, $S^{(1,1,1,1,1,1,1,1)}$ appears once, $S^{(2,1,1,1,1,1,1)}$ appears five times, and so on. This decomposition provides insight into the structure of the induced representation in terms of simpler irreducible representations.

$$Sym(T_4) \cong S_1 \times S_1 \times S_2 \wr S_2$$

$$Sym(T_5) \cong S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$$

$$Sym(T_6) \cong S_1 \times S_1 \wr S_2$$

$$Sym(T_7) \cong S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$$

$$Sym(T_8) \cong S_1 \times S_1 \times S_2 \wr S_2$$

$$Sym(T_9) \cong S_1 \times S_1 \times S_2 \times S_2 \times S_2$$

$$Sym(T_{10}) \cong S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$$

$$Sym(T_{11}) \cong S_1 \times S_1 \times S_2 \wr S_2$$

$$Sym(T_{12}) \cong S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$$

$$Sym(T_{13}) \cong S_1 \times S_1 \times S_2 \wr S_2$$

$$Sym(T_{14}) \cong S_1 \times S_1 \times S_2 \times S_2 \times S_2$$

$$Sym(T_{15}) \cong S_2 \times S_2 \wr S_2$$

$$Sym(T_{16}) \cong S_1 \times S_1 \times S_2 \times S_2 \times S_2$$

$$Sym(T_{17}) \cong S_2 \wr S_2$$

$$Sym(T_{18}) \cong S_1 \times S_1 \times S_1 \times S_2 \times S_1 \times S_2$$

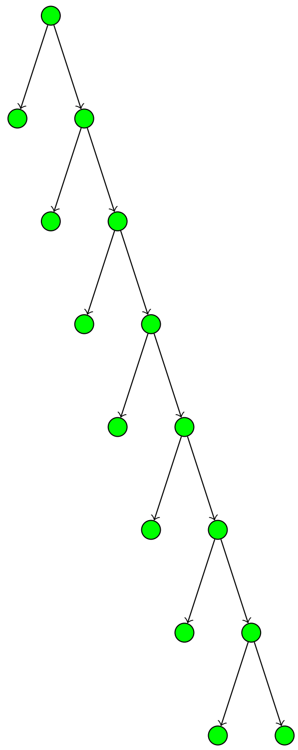
$$Sym(T_{19}) \cong S_1 \times S_1 \times S_2 \wr S_2$$

$$Sym(T_{20}) \cong S_1 \times S_1 \times S_2 \times S_2 \times S_2$$

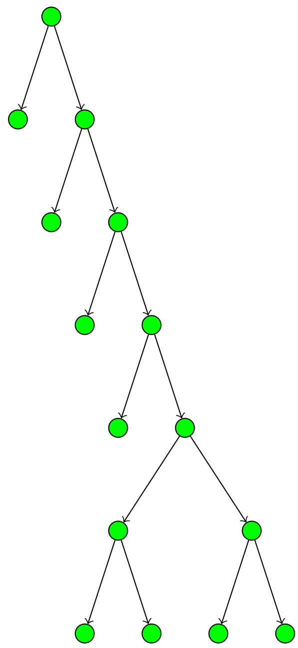
$$Sym(T_{21}) \cong S_1 \times S_1 \times S_2 \wr S_2$$

In the above, $\times$ is used for the direct product, $\circ$ represents the plethysm operation.

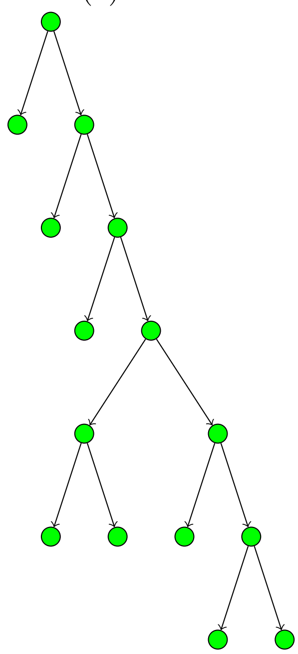
Binary trees corresponding to each P_n^k



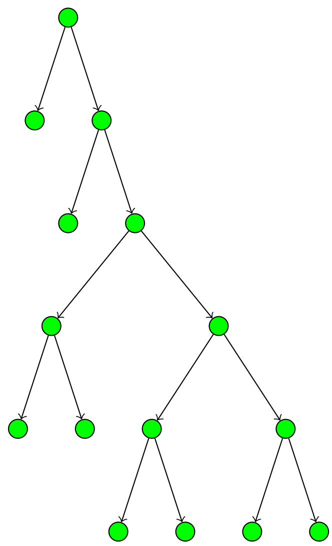
(a) Tree 1



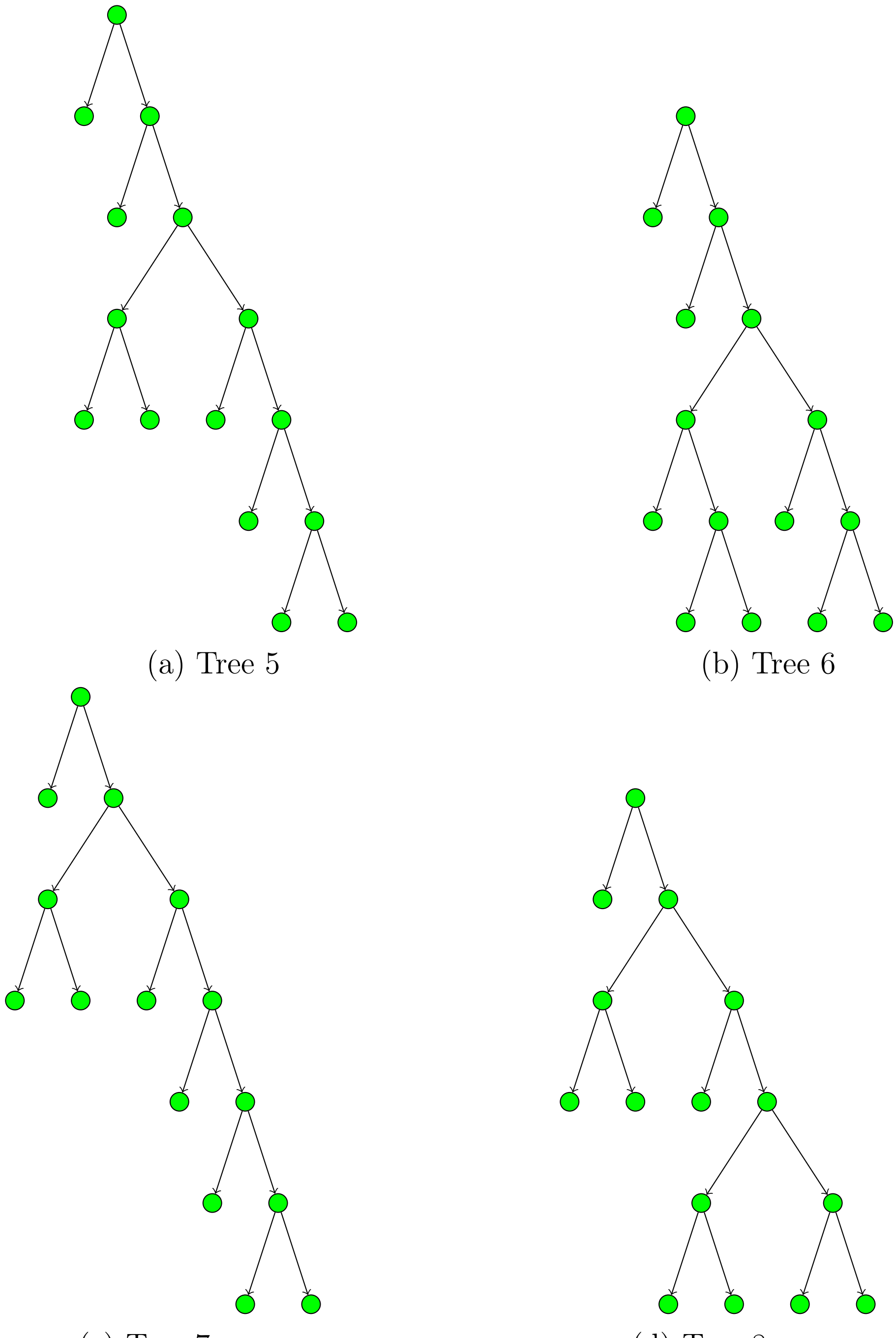
(b) Tree 2



(c) Tree 3

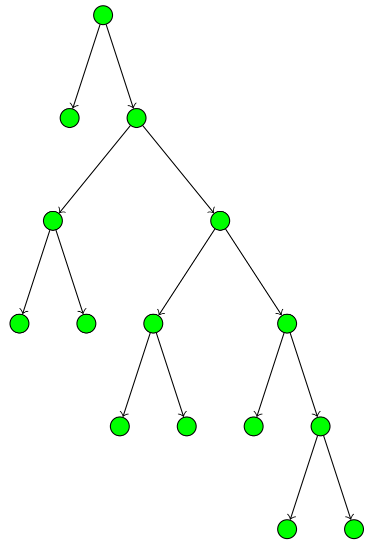


(d) Tree 4

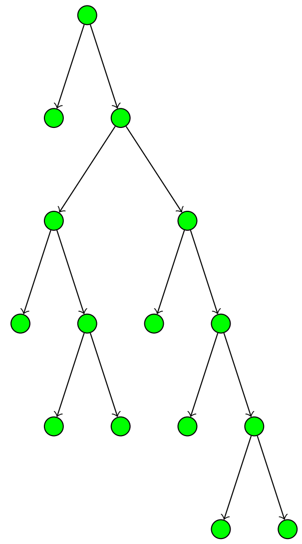


(a) Tree 5 (b) Tree 6

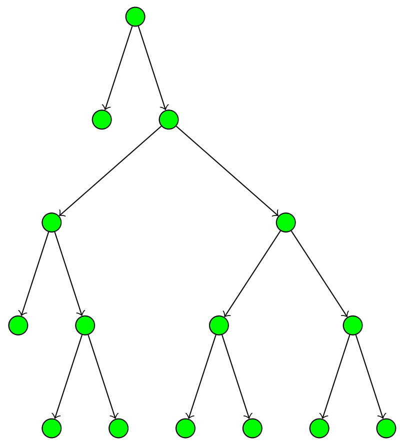
(c) Tree 7 (d) Tree 8



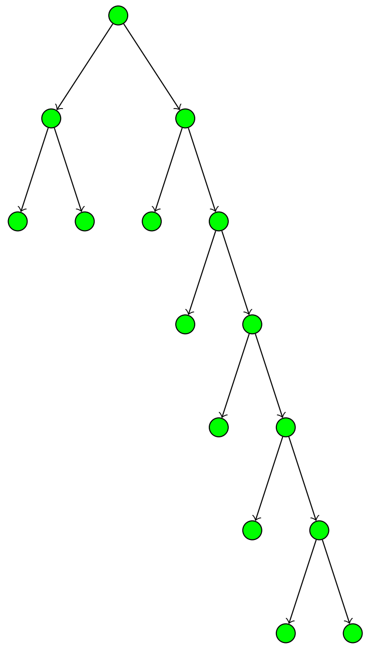
(a) Tree 9



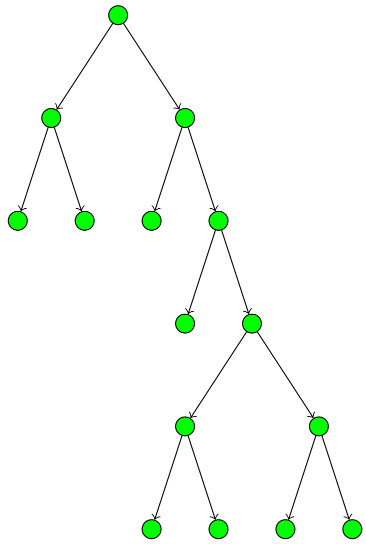
(b) Tree 10

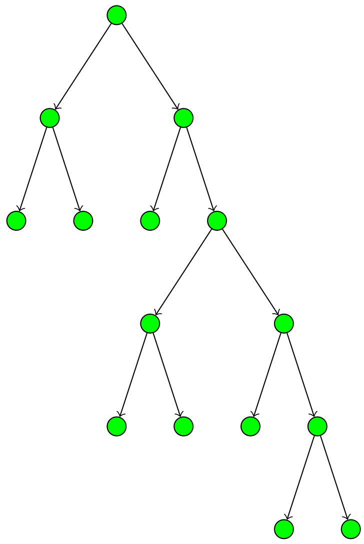


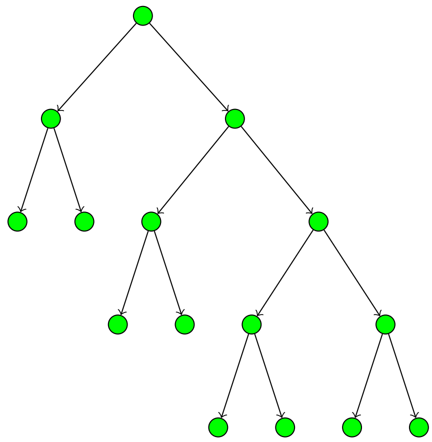
(c) Tree 11

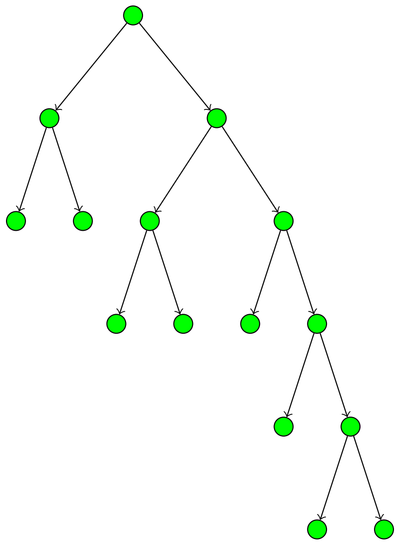


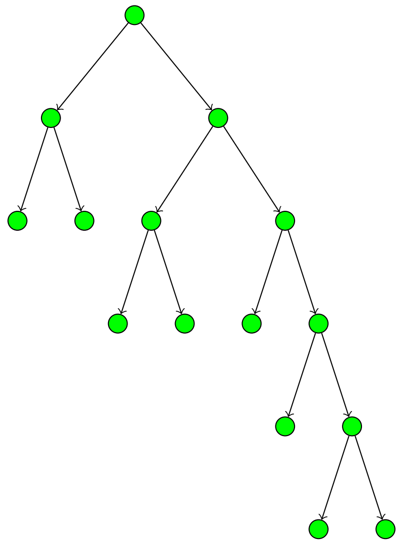
(d) Tree 12


(a) Tree 13

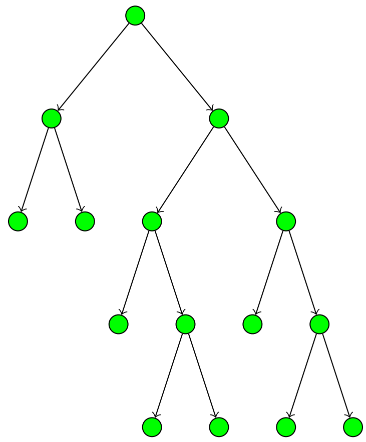

(b) Tree 14


(c) Tree 15

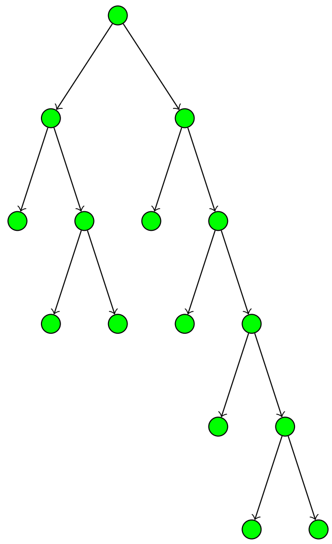

(d) Tree 16



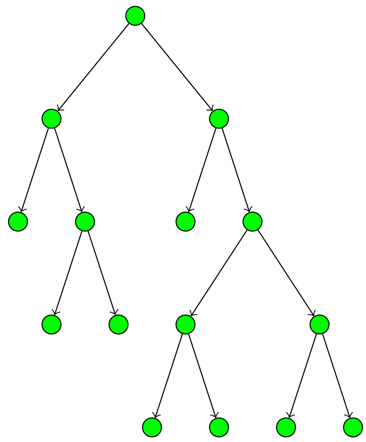
(a) Tree 16



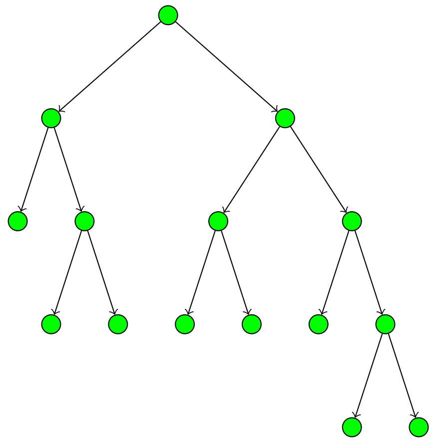
(b) Tree 17



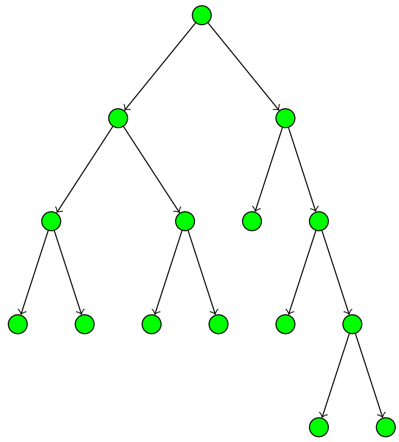
(c) Tree 18



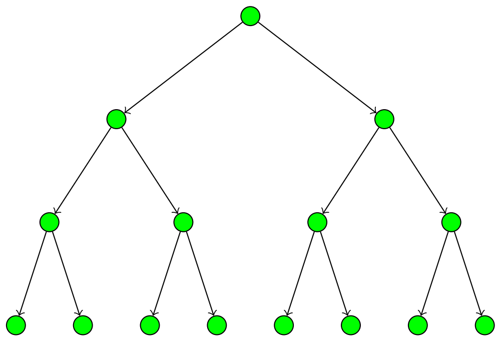
(d) Tree 19



(a) Tree 20



(b) Tree 21



(a) Tree 22

Conclusion

In this thesis, we have explored the S_n-module structures of free anti-commutative algebras. We have provided a detailed theoretical background, defined key concepts, and illustrated our results with examples. Our study contributes to the understanding of the representation theory of symmetric groups and has potential applications in mathematical physics and other areas of mathematics.

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ҒЫЛЫМ ДЕПАРТАМЕНТІ

SCIENCE DEPARTMENT

05.06. 2024 ж. № 14-14/181

АНЫҚТАМА

Ғылым департаменті, Инженерлік және жаратылыстану ғылымдары факультетінің 2 курс магистранты Ыдырыс Әбдібек Қасымжанұлы «**Ss-module structures of free anti-commutative algebra**» атты мақалалары «СДУ Хабаршысының» 2024 - №2 (65), жарияланымына қабылдағандығын растайды.

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