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Numerical Methods for HJB Equations in Option Pricing with Greek Sensitivities

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Supervisor: **Rakhymzhan Kazbek**

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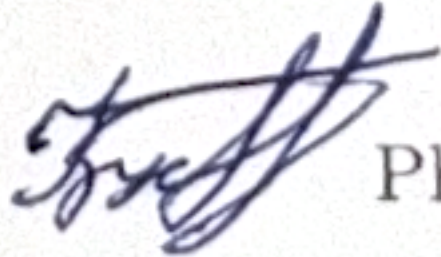
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DIPLOMA PROJECT

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7M05401 - "Mathematics"

Supervisor:  PhD, Rakhymzhan Kazbek

Student:  Kazhymurat Berikbayev

Kaskelen, 2026

Declaration

I confirm that this dissertation is my own original work, and that any materials from other sources are accompanied by proper and complete citations.

Kazhymurat Berikbayev

June 2026

Acknowledgements

I want to thank my supervisor Rakhymzhan Kazbek for all the help he gave me while I was working on this thesis. He was always available to answer my questions, no matter how basic they seemed, and explained every concept with clarity and patience. His feedback helped me fix my mistakes and think more carefully about the results. I am deeply grateful for his understanding and support from the very beginning to the end of this work.

I would also like to express my gratitude to my senior colleagues for their mentoring and assistance with academic matters throughout this work.

Dedication

This dissertation is dedicated to my family, and especially to my parents, for their unconditional love, support, and encouragement throughout my life and studies.

I also dedicate this work to my mathematics teachers at school and at university, who sparked my interest in the subject and laid the foundation for everything in this thesis.

Abstract

Usually, optimal control problems in financial mathematics are solved by maximizing profit and using the Hamilton-Jacobi-Bellman equation. This leads to nonlinear PDE, the solution of which is based on numerical methods.

In this paper, we use the method described in the article by Christara and Wu: second-order finite differences on a non-uniform grid. However, instead of the long position discussed there, we study the short position. Therefore, here too we use the same time step, calculated using the Crank-Nicolson-Rannacher method, and we employ an iterative solver with a penalty term. The nonuniform grid puts more nodes near the strike price. This is important for keeping second-order convergence when the straddle payoff is not smooth. Rannacher smoothing at the first time steps helps remove unwanted oscillations from the kink in the initial condition. The nonlinear penalty-like term comes from the optimal control in the stock borrowing fee model. We solve it at each time step with a Newton-type iterative algorithm that uses two stopping criteria.

Our numerical tests show secondorder convergence for the short position. The results agree with the benchmarks of Forsyth and Labahn and also Christara and Wu. The main result of this work is the numerical calculation of the Greeks for this nonlinear HJB model: Delta, Gamma, and Theta. As far as we know, there is no other full numerical study of the Greeks for the short position using finite differences on HJB problems. This is the main applied result of our work.

Аңдатпа

Әдетте қаржылық математикадағы оңтайлы басқару есептері пайданы барынша арттыру және Гамильтон–Якоби–Беллман теңдеуін пайдалану арқылы шешіледі. Бұл сызықтық емес дербес туындылы дифференциалдық теңдеуге әкеледі, оның шешімі сандық әдістерге негізделеді.

Осы мақалада біз Христара мен Ву мақаласында сипатталған әдісті, яғни біркелкі емес тордағы екінші ретті шекті айырмалар әдісін қолданамыз. Алайда аталған жұмыста қарастырылған long position орнына, біз short position жағдайын зерттейміз. Сондықтан бұл жұмыста да біз Кранк–Николсон–Ранначер әдісі бойынша анықталған уақыт қадамын қолданамыз және жазалау мүшесі бар итерациялық шешу әдісін пайдаланамыз. Біркелкі емес тор страйк бағасы маңында түйіндердің көбірек шоғырлануын қамтамасыз етеді. Straddle төлем функциясы тегіс болмаған жағдайда екінші ретті жинақталу дәлдігін сақтау үшін бұл маңызды. Уақыттың алғашқы қадамдарында қолданылатын Ранначер тегістеуі бастапқы шарттағы сыну нүктесінен туындайтын қажетсіз тербелістерді жоюға көмектеседі. Сызықтық емес айыппұл тәрізді мүше акцияны қарызға алу төлемі моделіндегі оңтайлы басқарудан туындайды. Оны әрбір уақыт қадамында екі тоқтату критерийіне негізделген Ньютон типті итерациялық алгоритм арқылы шешеміз.

Жүргізілген сандық тәжірибелер қысқа позиция үшін екінші ретті жинақталуды растайды. Алынған нәтижелер Форсайт пен Лабанның, сондай-ақ Христара мен Вудың эталондық нәтижелерімен жақсы сәйкестік көрсетеді. Осы жұмыстың негізгі нәтижесі – осы сызықтық емес HJB моделі үшін грек параметрлерін, атап айтқанда Delta, Gamma және Theta көрсеткіштерін сандық есептеу болып табылады. Біздің білуімізше, HJB теңдеулеріндегі қысқа позиция үшін шекті айырмалар әдісін қолдана отырып грек параметрлерін толық сандық зерттеуге арналған басқа жұмыс жоқ. Бұл біздің зерттеуіміздің негізгі қолданбалы нәтижесі болып табылады.

Abbreviations

BSM	Black-Scholes-Merton model
CN	Crank-Nicolson scheme
CNR	Crank-Nicolson-Rannacher procedure
FDM	Finite Difference Method
FEM	Finite Element Method
HJB	Hamilton-Jacobi-Bellman equation
$\mathcal{L}$	Differential operator
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SBF	Stock Borrowing Fee model
TDMA	Thomas (Tridiagonal Matrix) Algorithm

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Chapter 1

Background and motivations

1.1 Introduction

Option pricing is a key challenge in mathematical finance. The classical Black-Scholes model [1] is a well-known framework for valuing European options, but it relies on several assumptions that simplify reality, and these do not always match real market conditions. The Black-Scholes equation cannot directly account for situations where there are borrowing fees. For example, if a trader wants to open a short position early or on stocks. Because we now take into account the borrowing and lending fees, our equation, of course, is no longer linear. Instead, it transforms into the Hamilton-Jacobi-Bellman (HJB) equation, which is one of the fundamental PDE problems in optimal control. Below you can see difference between Black Scholes PDE and HJB equation, respectively:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \max\{\mathcal{L}_1 V, \mathcal{L}_2 V, 0\},$$

Here, we can choose the best possible hedging strategy at every point in time. For a short position, the main equation includes a max term, and for a long position, a min term, respectively. From a numerical perspective, the HJB equation cannot be solved using standard linear equation solvers. And the first to study these equations using numerical methods were Forsyth and Labahn [4]. They analyzed the SBF problem step by step and formulated it as the optimal control problem we know today. To solve it, they applied the policy iteration method [10] in combination with the Crank-Nicolson time step. Their results for the European straddle option are now widely accepted as the gold standard. My goal in this article, together with my supervisor Rakhymzhan Kazbek, was to reproduce and extend their convergence tables using Greeks. In 2002, Forsyth and Vetzal [5] introduced the concept of iteration with a penalty term when working with American options. As a result, this method also proved successful and is still used by modern mathematicians

today. Christara and Wu [2] improved upon earlier methods involving penalty terms [5, 11]. They found an alternative way to handle the nonlinear max and min terms in the FEM equations. They used triangular penalty matrices that fit easily into the standard finite difference scheme. Furthermore, they proposed two stopping criteria that work together—one based on the solution and the other on the penalty matrix.

In this work, we directly implement and extend the framework of Christara and Wu [2], working specifically with their PDE equation (9) for the short-term nonlinear version of the SBF problem:

$$\begin{cases} V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_l(SV_S - V) + \max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} \\ V(S, 0) = \max(S - K, K - S), \quad V(0, \tau) = K, \quad V(S_{\max}, \tau) = S_{\max} - K \end{cases}$$

Above, it is written what boundary conditions were given for this equation. Paper focused on the European straddle option with stock borrowing fees. We checked our numerical results against the reference results obtained using the finite difference method by Forsyth and Laban [4], and compared them with the results of the finite element method by Kazbek and Abdukarymova [7]. While Christara and Wu [2] mainly has results for a long position with a minimum term, our goal was to add a full study of the Greeks specifically for the short position. Because of this our paper tries to fill this small but nonetheless important gap in the existing literature. Therefore, we used the same methods and algorithms for MATLAB as in the paper: a second-order finite difference method on a nonuniform grid with Crank-Nicolson-Rannacher time stepping [9]. Of course, we also handled the nonlinear term using a penalty-type iteration with two stopping criteria. But the main small novelty was the implementation and analysis of the Greeks (Delta, Gamma, and Theta) on the short position.

Sure, we didn't immediately start studying the HJB equation and what optimal solutions we could use. To understand how the algorithm works and how the calculations were performed, we had to start with the Black-Scholes [1] equation. More specifically, we needed to discretize the linear equation and transform it into a tridiagonal form so that we could verify convergence using modern programming languages such as MATLAB. To do this, we relied on Hirsu [6]'s book, which provides a very detailed analysis of the computations and includes analytical background. Thanks to this book, it was clear how to correctly add the nonlinear part step by step and how to switch between solution methods. For example, we started with the explicit Euler method, then moved on to the implicit method, and later used the Crank-Nicolson-Rannacher method with implementing Greeks. It worked very well and allowed us to evaluate the convergence and the results.

The problem of nonlinear option pricing can be solved using other methods as well; one of them is the finite element method. Most recently, in 2024, my supervisor Kazbek, along with Erlangga, Amanbek and Wei [8], applied finite element methods to the pricing of convertible bonds using a penalty function-

based approach. They demonstrated that higher-order basis functions can improve both accuracy and computational efficiency. Building on this work, Kazbek and Abdukarimova [7] extended the method specifically for the SBF HJB problem, using quadratic Lagrangian elements on a non-uniform grid in combination with the Crank-Nicolson-Rannacher time stepping [9]. Since the parameters in our article are the same as theirs, and the calculation method is also the same, their article served as another point of comparison for the results. And it worked really well, even though the spatial discretization was different—in the end, we obtained nearly identical numerical values. This allows a direct comparison of the two methodologies in terms of convergence behavior and computational cost.

Even with the progress above, numerically finding the Greeks for the SBF-HJB model, especially for short positions, is still not fully solved. The Greeks are important for hedging in practice, because they show how the option price changes when the underlying asset price or time changes. In the nonlinear HJB case, the Greeks behave more complicated than in the usual Black-Scholes model. So, to compute them accurately, we need to handle the same numerical problems as in pricing. treatment of the same numerical difficulties that arise in the pricing problem itself. The This paper aims to fill this gap by using the finite difference method described above to compute and study the Greeks numerically.

Chapter 2

Historical Background

This chapter traces the origins of the main mathematical tools used in this thesis, history of options and models: finite difference approximations on non-uniform grids, the matrix representation of differential operators, penalty methods for constrained problems, and time-stepping schemes with initial smoothing and etc.. Our goal is to give the reader full enough context to understand where each idea came from and why it took the form it did, before those ideas are applied in the chapters that follow.

2.1 A Brief History of Options

Options, in their most basic form, are contracts that give one party the right but not the obligation to buy or sell an asset at a predetermined price on or before a specified date. The idea is far older than modern financial mathematics. Aristotle, in *Politics* (circa 350 BC), recounts how the philosopher Thales of Miletus secured the right to use olive presses at harvest time by paying a deposit in advance, profiting when the harvest proved large and demand for presses was high. In the 17th century, options on tulip bulbs were traded during the Dutch tulip mania of 1636-1637, and options on shares appeared on the Amsterdam Stock Exchange shortly after its founding in 1602. By the 19th century, options on commodities and stocks were traded in London and New York, though without any agreed method for determining what a fair price should be.

The absence of a pricing theory meant that options were priced by negotiation and custom. The first serious mathematical attempt came from Louis Bachelier in his 1900 doctoral thesis *Théorie de la Spéculation*, supervised by Henri Poincaré at the Sorbonne. Bachelier modelled the stock price as a random walk with normally distributed increments - arithmetic Brownian motion - and derived a formula for the price of a European call option. His work was largely ignored until it was rediscovered in the 1950s and 1960s by economists working on stochastic processes.

The modern theory began with Fischer Black, Myron Scholes, and Robert Merton in 1973. Black and Scholes showed that under assumptions of continuous

trading, no arbitrage, constant volatility, and a single risk-free rate, the price of a European option satisfies a partial differential equation whose solution for a call option is the closed-form formula

$$C = S_0 N(d_1) - K e^{-rT} N(d_2),$$

where $N(\cdot)$ is the standard normal cumulative distribution function and d_1, d_2 are expressions involving the current price S_0 , strike K , rate r , volatility σ , and maturity T . Merton extended the framework to continuous-time hedging and introduced the replicating portfolio argument that underpins all of derivatives pricing. Scholes and Merton received the Nobel Prize in Economics in 1997.

The Black-Scholes-Merton model was a significant step, but its assumptions are restrictive. Observed market prices deviate from its predictions in a consistent pattern: implied volatility varies with strike and maturity, forming a shape known as the volatility smile. This gap between model and market prompted a sequence of generalisations over the following decades.

Merton (1976) introduced jump-diffusion models, adding a Poisson jump process to the stock price to capture sudden large movements that a continuous diffusion cannot reproduce. Heston (1993) proposed a stochastic volatility model in which volatility itself follows a mean-reverting process:

$$dv = \kappa(\theta - v) dt + \xi \sqrt{v} dW_2,$$

correlated with the stock price process. The Heston model reproduces the volatility smile and admits a semi-closed-form solution via characteristic functions. Dupire (1994) and Derman and Kani (1994) independently developed local volatility models, in which volatility is a deterministic function $\sigma(S, t)$ calibrated directly to observed market prices. Madan, Carr, and Chang (1998) proposed the Variance Gamma model, replacing Brownian motion with a process of finite variation to capture skewness and excess kurtosis in returns. Each of these models requires numerical methods for most payoff structures, since closed-form solutions exist only in special cases.

The model in this thesis comes from a different direction: the effect of market frictions on the pricing equation itself. When borrowing and lending rates differ, or when selling a stock short requires paying a borrowing fee, the replication argument produces a nonlinear PDE rather than the linear Black-Scholes equation. Bergman (1995) showed that differential rates lead to an equation where the interest rate in the operator depends on the sign of the hedging position. The specific formulation studied here - incorporating a stock borrowing fee r_f alongside separate borrowing rate r_b and lending rate r_l - leads to a Hamilton-Jacobi-Bellman equation. Solving this nonlinear equation numerically for a short straddle position, using finite differences and a penalty iteration, is the subject of this thesis.

2.2 From Brownian motion to BSM

All of this started with an observation that at first glance had nothing to do with finance. In 1827, the Scottish botanist Robert Brown noticed that pollen grains suspended in water move chaotically and unpredictably. He could not explain the reason - he simply recorded the fact. Nearly eight decades passed before Albert Einstein gave this phenomenon a rigorous mathematical description in 1905, showing that such motion is caused by continuous random collisions with molecules of the surrounding medium. Norbert Wiener formalized this in 1923 as a mathematical process that today bears his name — the Wiener process W_t , or standard Brownian motion. Its key properties are: $W_0 = 0$, increments are independent and normally distributed, $W_t - W_s \sim \mathcal{N}(0, t - s)$ for $t > s$.

One might ask: what does this have to do with finance? Yet it was precisely this idea of random walk that became the foundation of modern asset pricing theory. Louis Bachelier, as early as 1900 in his doctoral dissertation, suggested that stock prices behave like Brownian motion. His work went almost unnoticed at the time, but half a century later the idea was rediscovered and developed further.

The next fundamental step was Itô's lemma, formulated by the Japanese mathematician Kiyosi Itô in 1944. It allows one to work with functions of stochastic processes - something classical calculus cannot handle directly. If the asset price S is described by the stochastic differential equation

$$dS = \mu S dt + \sigma S dW_t, \quad (2.2.1)$$

this is called geometric Brownian motion (GBM). Here μ is the drift (expected return), σ is the volatility, and dW_t is the random noise term. The model is convenient because the price always remains positive, which is physically reasonable for stocks.

Now let $V(S, t)$ denote the option value depending on the asset price S and time t . By Itô's lemma:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2. \quad (2.2.2)$$

Substituting dS and using the rule $(dW_t)^2 = dt$, one obtains:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t. \quad (2.2.3)$$

Then in 1973, Fischer Black and Myron Scholes published a paper that effectively created the modern derivatives market. Their idea was simple and brilliant at the same time: if one constructs a portfolio Π consisting of the option and a certain number of shares chosen so that the portfolio does not depend on the random term dW_t , then such a portfolio must grow at the risk-free rate - otherwise an arbitrage

opportunity arises. Constructing the delta-neutral portfolio:

$$\Pi = V - \frac{\partial V}{\partial S} \cdot S, \quad (2.2.4)$$

its change is:

$$d\Pi = dV - \frac{\partial V}{\partial S} dS = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.2.5)$$

The random component has vanished. By the no-arbitrage principle $d\Pi = r\Pi dt$, which immediately yields the Black-Scholes equation (BSE):

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (2.2.6)$$

This is a linear second-order partial differential equation (PDE). For a European call option it admits an analytical solution - the celebrated Black-Scholes formula. Scholes and Merton received the Nobel Prize in Economics in 1997 for this work.

However, the model has serious limitations that practitioners feel in practice. One of the main ones is the assumption that borrowing and lending occur at the same risk-free rate r . In reality the borrowing rate r_b exceeds the lending rate r_l , and opening a short position requires paying an additional stock borrowing fee r_f . The classical model simply ignores these real-world costs, and it is precisely here that the story of the Hamilton–Jacobi–Bellman (HJB) equation begins.

2.3 The Hamilton-Jacobi-Bellman Equation

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \max\{\mathcal{L}_1 V, \mathcal{L}_2 V, 0\},$$

The HJB equation arises naturally when the trader selects an optimal hedging strategy at every instant from several possible regimes. Mathematically this is expressed through the max operator (for a short position) or min (for a long position). Unlike the linear BSE, no closed-form solution exists in general, and one must resort to numerical methods. Forsyth and Labahn [4] systematically formulated the stock borrowing fee (SBF) problem as an optimal control problem, obtaining for the short position:

$$\mathcal{L}(r, \sigma)V \equiv \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV.$$

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_l(SV_S - V) + \max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\},$$

where $\tau = T - t$ is the reversed time variable, $V_S = \partial V / \partial S$, and $V_{SS} = \partial^2 V / \partial S^2$.

The three arguments inside $\max$ correspond to three regimes: borrowing at rate r_b , lending with fee r_f , and the neutral position. The nonlinearity of the $\max$ operator means that standard linear solvers are not applicable — dedicated iterative procedures are required.

2.4 Finite Differences: From Newton to Non-Uniform Grids

Finite difference approximations of derivatives predate numerical computing by more than two centuries. Isaac Newton introduced what he called the method of differences in his *Methodus Differentialis* (published posthumously in 1711), where he used the forward difference operator $\Delta f_i = f_{i+1} - f_i$ to construct interpolating polynomials from tabulated values. James Gregory had reached similar conclusions independently around 1670. The Newton-Gregory forward difference formula,

$$f(x) = f_0 + \binom{s}{1} \Delta f_0 + \binom{s}{2} \Delta^2 f_0 + \cdots, \quad s = \frac{x - x_0}{h},$$

was the standard tool for interpolation and differentiation for roughly two centuries after its discovery.

The connection between Taylor series and finite differences was made explicit by Brook Taylor in *Methodus Incrementorum* (1715). Writing out the expansion

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

and manipulating the expressions for $f(x + h)$ and $f(x - h)$ together immediately gives the centred first-derivative approximation

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} + O(h^2)$$

and the centred second-derivative approximation

$$f''(x) \approx \frac{f(x + h) - 2f(x) + f(x - h))}{h^2} + O(h^2).$$

Gauss recognised that centred formulas are more accurate than one-sided ones and used symmetric difference tables in his geodetic calculations throughout the early 19th century.

These formulas all assumed a uniform grid with fixed step h . The extension to non-uniform grids became necessary in computational fluid dynamics during the 1950s and 1960s, where boundary layers near walls require very fine spacing while the interior of the domain can be coarser. On a grid where the left spacing is $h_i^- = x_i - x_{i-1}$ and the right spacing is $h_i^+ = x_{i+1} - x_i$, two separate Taylor

expansions give

$$\begin{aligned} f_{i+1} &= f_i + h_i^+ f'_i + \frac{(h_i^+)^2}{2} f''_i + O((h_i^+)^3), \\ f_{i-1} &= f_i - h_i^- f'_i + \frac{(h_i^-)^2}{2} f''_i + O((h_i^-)^3). \end{aligned}$$

Eliminating f''_i from the first pair and f'_i from the second yields second-order centred approximations for f'_i and f''_i on the non-uniform grid. These formulas reduce to the classical ones when $h_i^- = h_i^+ = h$. They appear in the CFD monographs of Patankar (1980) and Anderson, Tannehill, and Pletcher (1984), though the algebra is elementary enough to have been known earlier. When the formulas are written out for all interior nodes simultaneously, the result is a tridiagonal matrix - a structure that could not be exploited computationally until the direct solver described in Section 2.8 was available.

2.4.1 FDM of Derivatives

Once the grid is fixed, derivatives are approximated by finite differences. On a non-uniform grid the second-order centred approximations at an interior node S_i are:

$$\begin{aligned} V_S|_i &\approx \frac{-(h_i^+)^2 V_{i-1} + ((h_i^+)^2 - (h_i^-)^2) V_i + (h_i^-)^2 V_{i+1}}{h_i^- h_i^+ (h_i^- + h_i^+)}, \\ V_{SS}|_i &\approx \frac{2}{h_i^- + h_i^+} \left(\frac{V_{i+1} - V_i}{h_i^+} - \frac{V_i - V_{i-1}}{h_i^-} \right). \end{aligned}$$

Assembling these expressions for all interior nodes $i = 1, \dots, N-1$ yields tridiagonal matrices D_1 and D_2 of size $(N-1) \times (N-1)$ - the first and second derivative matrices respectively. Boundary contributions are handled separately and moved to the right-hand side of the linear system.

2.4.2 Uniform and Non-Uniform grids

Before discretizing the equation one must decide how to distribute nodes over the domain $[0, S_{\max}]$. The simplest option is a uniform grid with step $h = S_{\max}/N$, where N is the number of intervals, giving nodes $S_i = ih$ for $i = 0, 1, \dots, N$. This is straightforward to implement but inefficient: the option price changes most rapidly near the strike K , while it is nearly linear near the boundaries. Spending the same node density everywhere is wasteful.

In practice one therefore uses non-uniform grids that are refined near K . A standard approach is a coordinate transformation. One introduces an auxiliary variable $\xi \in [0, 1]$ and constructs a mapping $S(\xi)$ so that nodes concentrate where the solution varies rapidly. A common choice is the hyperbolic-sine transformation:

$$S_i = K + c_1 \sinh\left(\frac{i}{N} c_2 - c_3\right), \quad i = 0, \dots, N,$$

where the constants c_1, c_2, c_3 are determined from the conditions $S_0 = 0$, $S_N =$

$S_{\max}$, and the desired node density near K . The local steps $h_i^- = S_i - S_{i-1}$ and $h_i^+ = S_{i+1} - S_i$ are now variable, and this must be accounted for when constructing finite-difference operators.

2.5 Matrix Formulation of Differential Operators

The idea that a differential operator could be replaced by a matrix acting on a vector of function values at grid points developed gradually, and from more than one direction.

One line of development came from the calculus of variations. Walter Ritz (1909) proposed approximating the solution of a boundary value problem as a finite linear combination of known basis functions, reducing the problem to a linear system whose size equals the number of basis functions. Richard Courant (1943) extended this by subdividing the domain into triangles and using piecewise linear functions as the basis, which is the geometric origin of the finite element method. In parallel, Collatz (1933) and colleagues in Germany were replacing differential equations directly with difference equations over a grid, obtaining linear systems whose coefficient matrices encoded the structure of the underlying operator. These two traditions - variational and algebraic - were pursuing the same goal through different means.

The step of treating the coefficient matrix as an object of study in its own right became natural with the arrival of electronic computers after World War II. Von Neumann and Goldstine (1947) introduced stability analysis based on the eigenvalues of the iteration matrix, making the connection between the spectrum of the operator and the behaviour of the numerical method explicit and precise. Courant, Friedrichs, and Lewy had already identified in 1928, well before computers existed, a necessary condition for convergence that linked the time step, the spatial step, and the propagation speed of the equation - the CFL condition - and this condition is naturally expressed in terms of the spectral radius of the update matrix.

Through the 1950s and 1960s, as numerical analysis became an academic discipline, it became standard to classify the matrices arising from discretised elliptic operators by their algebraic properties: symmetry, diagonal dominance, positive definiteness, sparsity. Varga's *Matrix Iterative Analysis* (1962) and Forsythe and Wasow's *Finite Difference Methods for Partial Differential Equations* (1960) codified these properties and linked them to the convergence of iterative solvers. Strang and Fix's *An Analysis of the Finite Element Method* (1973) did the same for the variational side, and over time the two communities adopted a shared algebraic vocabulary. The compact matrix notation used throughout this thesis - derivative matrices D_1 , D_2 multiplied by diagonal coefficient matrices - is a direct product of this tradition.

2.6 Penalty Methods and Variational Inequalities

Problems in which a solution must satisfy both a differential equation and an inequality constraint arise in mechanics, economics, and engineering. The mathematical tools for handling such problems have a long history.

The method of Lagrange multipliers, introduced in *Mécanique Analytique* (1788), was the first systematic way to convert a constrained problem into an unconstrained one. A multiplier variable is introduced for each constraint, and the enlarged system is solved for both the original unknowns and the multipliers. The approach is exact but increases the size of the problem. Courant (1943) proposed a different strategy: instead of enforcing the constraint $g(x) \leq 0$ exactly, add a penalisation term to the objective function,

$$f(x) + \frac{1}{\varepsilon} \max(g(x), 0)^2,$$

and solve the unconstrained problem, letting $\varepsilon \rightarrow 0$. The constraint is satisfied only approximately, but the system size does not grow, and standard solvers can be applied. Courant introduced this idea in the context of boundary conditions and equilibrium problems in continuum mechanics.

The rigorous functional-analytic theory came from Guido Stampacchia and Jacques-Louis Lions in the 1960s. Their framework of variational inequalities - find u in a convex set K such that

$$a(u, v - u) \geq \langle f, v - u \rangle \quad \text{for all } v \in K$$

- gave existence, uniqueness, and regularity results for a broad class of constrained problems, including contact mechanics, seepage through dams, and obstacle problems. The connection to discrete linear complementarity problems (LCP) of the form

$$w = Mz + q \geq 0, \quad z \geq 0, \quad z^\top w = 0$$

was clarified by Cottle and Dantzig (1968) and Lemke (1965), who showed that the two formulations are equivalent in finite dimensions. This opened the way for algorithms from operations research - pivoting methods, interior point methods, splitting iterations - to be applied to discretised inequality problems.

The iterative penalty approach, in which a sequence of linear systems is solved with the penalty matrix updated at each step rather than solving the full LCP at once, was studied in the context of finite element obstacle problems by Glowinski, Lions, and Trémolières (1981). They proved convergence of the iterates and showed that the active set - the set of nodes where the constraint is binding - stabilises after finitely many iterations under monotonicity conditions on the operator. This stabilisation property is what makes it possible to use the change in the penalty matrix itself as a stopping criterion, rather than relying solely on the change in the iterates.

2.6.1 Derivative Matrices and Base Operators

Let $\mathbf{v} = (V_1, \dots, V_M)^\top$ be the vector of solution values at interior nodes. Here by using the nonuniform finite difference matrices T_1 and T_2 assembled in the previous section, the first and second spatial derivatives are approximated as:

$$\frac{\partial V}{\partial S} \approx T_1 \mathbf{v} + \mathbf{b}_{c_1}, \quad \frac{\partial^2 V}{\partial S^2} \approx T_2 \mathbf{v} + \mathbf{b}_{c_2},$$

where $\mathbf{b}_{c_1}$ and $\mathbf{b}_{c_2}$ are boundary contribution vectors arising from the Dirichlet conditions at $S = 0$ and $S = S_{\max}$.

The linear part of the HJB equation is then written compactly as:

$$L\mathbf{v} = \frac{\sigma^2}{2} D_{S^2} T_2 \mathbf{v} + r_l (D_S T_1 - I) \mathbf{v},$$

where $I \in \mathbb{R}^{M \times M}$ is the identity matrix, and D_S , D_{S^2} are the diagonal matrices (full view in Methodology section)

$$D_S = \text{diag}(S_1, \dots, S_M), \quad D_{S^2} = \text{diag}(S_1^2, \dots, S_M^2).$$

The operator L represents the linear part of the equation, corresponding to the standard diffusion-advection-reaction structure. Since T_1 and T_2 are tridiagonal and D_S , D_{S^2} are diagonal, the product $D_{S^2} T_2$ and $D_S T_1$ remain tridiagonal, so L itself is a tridiagonal operator. The full system matrix is therefore:

$$A = \frac{\sigma^2}{2} D_{S^2} T_2 + r_l (D_S T_1 - I),$$

which is the base operator matrix used at every time step. The nonlinear max term in the HJB equation is treated separately through the penalty matrices $P(\mathbf{v})$, described in the next section.

2.6.2 Penalty Matrices

The key idea of Christara and Wu [2] is to replace the max operator with a penalty term added iteratively to the linear system. At each interior node i one of three regimes is active:

- **Regime 1:** $(r_b - r_l)(S_i V_S|_i - V_i) \geq \max(-r_f S_i V_S|_i, 0)$;
- **Regime 2:** $-r_f S_i V_S|_i \geq \max((r_b - r_l)(S_i V_S|_i - V_i), 0)$;
- **Regime 0:** both expressions are non-positive, so $\max = 0$.

The penalty matrix P is triangular, and at each node it adds the contribution corresponding to the active regime. On iteration k the regime is determined from the current approximation $\mathbf{v}^{(k)}$, after which the system is solved with the updated matrix. Two stopping criteria operate jointly: the first checks the change in the solution, $\|\mathbf{v}^{(k+1)} - \mathbf{v}^{(k)}\| < \varepsilon_1$, and the second checks the change in the penalty

matrix itself, i.e. the stabilisation of the active regimes. Iteration stops only when both conditions are satisfied simultaneously.

2.7 Time Discretisation: From Euler to Crank-Nicolson-Rannacher

Euler's Method (1768)

Leonhard Euler introduced the first numerical method for ordinary differential equations in *Institutionum Calculi Integralis* (1768). For the equation $u' = f(t, u)$, he proposed stepping forward by

$$u^{n+1} = u^n + \Delta t f(t^n, u^n),$$

using only information at the current time level. This explicit, or forward, Euler method is first-order accurate and stable only when $\Delta t \leq Ch^2$ for parabolic problems, a constraint that becomes impractical on fine grids. The implicit, or backward, Euler method,

$$u^{n+1} = u^n + \Delta t f(t^{n+1}, u^{n+1}),$$

is also first-order accurate but unconditionally stable for parabolic problems, meaning no restriction on the step size is needed. The trade-off is that a system of equations must be solved at every step.

Higher-Order Methods: Adams, Runge, Kutta

During the 19th and early 20th centuries, the main effort in time integration was directed toward higher-order accuracy. Adams (1855) and Bashforth and Adams (1883) developed multistep methods that reuse function evaluations from previous steps. Runge (1895) and Kutta (1901) developed single-step methods that achieve high order by evaluating f at intermediate points within each step. These methods reach $O(\Delta t^p)$ accuracy for $p = 2, 3, 4$ and higher, and they remain the standard tools for ODE systems. For PDEs, the stability constraints of explicit high-order methods and the computational cost of implicit ones pushed practitioners toward second-order methods that balance these competing demands. Below were shown the main high-order numerical methods:

Adams-Bashforth two-step (explicit):

$$u^{n+1} = u^n + \frac{\Delta t}{2} (3f(t^n, u^n) - f(t^{n-1}, u^{n-1}))$$

Adams-Moulton two-step (implicit):

$$u^{n+1} = u^n + \frac{\Delta t}{2} (f(t^{n+1}, u^{n+1}) + f(t^n, u^n))$$

Runge-Kutta fourth-order (RK4):

$$u^{n+1} = u^n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(t^n, u^n),$$

$$k_2 = f\left(t^n + \frac{\Delta t}{2}, u^n + \frac{\Delta t}{2} k_1\right),$$

$$k_3 = f\left(t^n + \frac{\Delta t}{2}, u^n + \frac{\Delta t}{2} k_2\right),$$

$$k_4 = f(t^n + \Delta t, u^n + \Delta t k_3).$$

The Crank-Nicolson Scheme (1947)

For parabolic PDEs, the method that resolved the accuracy-stability tension was introduced by John Crank and Phyllis Nicolson in a 1947 paper in the *Proceedings of the Cambridge Philosophical Society*. Their idea was to average the explicit and implicit Euler discretisations:

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2} L u^{n+1} + \frac{1}{2} L u^n,$$

where L is the spatial operator. This gives second-order accuracy in time, $O(\Delta t^2)$, while remaining unconditionally stable for linear parabolic problems. The same formula arises from applying the trapezoidal quadrature rule to the time integral of Lu , which is why it inherits that rule's second-order accuracy. Lax and Richtmyer (1956) later gave a rigorous stability and convergence analysis using the equivalence theorem: for consistent linear schemes, stability is both necessary and sufficient for convergence.

The Rannacher Smoothing Procedure (1984)

The Crank-Nicolson scheme has one well-documented weakness. Its amplification factor - the factor by which each frequency component of the error is multiplied at each time step - is

$$g(\xi) = \frac{1 + \frac{\Delta t}{2} \lambda(\xi)}{1 - \frac{\Delta t}{2} \lambda(\xi)},$$

where $\lambda(\xi)$ are the eigenvalues of the spatial operator. For parabolic problems $\lambda(\xi) < 0$, so $|g(\xi)| = 1$ for all frequencies. The scheme is stable, but it damps nothing - every frequency component of the initial error is carried forward unchanged in magnitude. When the initial condition is not smooth, it contains high-frequency components that persist for many time steps and appear as oscillations in the numerical solution. The fully implicit Euler scheme has amplification factor

$$g(\xi) = \frac{1}{1 - \Delta t \lambda(\xi)},$$

which tends to zero as $|\lambda(\xi)| \rightarrow \infty$, meaning high-frequency components are strongly suppressed at each step.

Rolf Rannacher, in a 1984 paper in *Numerische Mathematik*, proposed replacing only the first few time steps with fully implicit Euler steps to damp the high-frequency components before they can pollute the rest of the solution, then switching to Crank-Nicolson for the remaining steps. Since the implicit steps act only near $t = 0$ where the initial transient is concentrated, their $O(\Delta t)$ error contributes only a low-order term to the global error, which is dominated by the $O(\Delta t^2)$ Crank-Nicolson error elsewhere. The combined scheme therefore retains second-order global accuracy while removing the oscillations. The θ -family formulation captures both schemes in a single expression:

$$\frac{u^{n+1} - u^n}{\Delta t} = \theta Lu^{n+1} + (1 - \theta) Lu^n, \quad \theta \in [0, 1],$$

with $\theta = 1$ giving fully implicit Euler and $\theta = \frac{1}{2}$ giving Crank-Nicolson. The Crank-Nicolson-Rannacher procedure sets $\theta = 1$ for the first two steps and $\theta = \frac{1}{2}$ for all subsequent steps. Giles and Carter (2006) showed that two implicit steps are sufficient: the high-frequency error after two such steps is already $O(\Delta t^2)$, matching the background Crank-Nicolson error, so further implicit steps provide no additional benefit.

2.7.1 Discretization of HJB equation

For time discretization the Crank-Nicolson-Rannacher (CNR) scheme is used. The standard Crank-Nicolson (CN) scheme, which is second-order accurate in time, reads:

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta \tau} = \frac{1}{2} \mathcal{L} \mathbf{v}^{n+1} + \frac{1}{2} \mathcal{L} \mathbf{v}^n + (\text{nonlinear part}),$$

which after rearrangement gives the linear system:

$$\left(I - \frac{\Delta \tau}{2} \mathcal{L} \right) \mathbf{v}^{n+1} = \left(I + \frac{\Delta \tau}{2} \mathcal{L} \right) \mathbf{v}^n + \Delta \tau \mathbf{f}^n,$$

where $\mathbf{f}^n$ includes boundary contributions and the nonlinear penalty term. Rannacher [9] proposed performing the first two time steps with the fully implicit Euler scheme in order to damp the oscillations caused by the non-differentiability of the initial condition (the straddle payoff has a kink at $S = K$). After that one switches to the standard CN scheme. This approach preserves second-order global accuracy while eliminating initial-layer artifacts. The resulting θ -scheme unifies both cases:

$$(I - \theta \Delta \tau \mathcal{L}) \mathbf{v}^{n+1} = (I + (1 - \theta) \Delta \tau \mathcal{L}) \mathbf{v}^n + \Delta \tau \mathbf{f}^n,$$

$$\theta_n = \begin{cases} 1, & n = 1, 2, \\ \frac{1}{2}, & n \geq 3. \end{cases}$$

The choice of two implicit steps is not arbitrary: one implicit step is insufficient to reduce the high-frequency error to the level of the Crank-Nicolson background error, while three or more steps provide no additional benefit. By the way, two

steps represent the minimal number for which the initial-layer contribution to the global error is of order $O(\Delta\tau^2)$, matching the dominant CN error and leaving the overall convergence rate unaffected.

2.8 The Thomas Algorithm and Tridiagonal Systems

Discretising a one-dimensional boundary value problem by finite differences produces a linear system in which each equation involves at most three consecutive unknowns:

$$a_i u_{i-1} + b_i u_i + c_i u_{i+1} = d_i, \quad i = 1, \dots, N.$$

The coefficient matrix of such a system is tridiagonal. Solving it by standard Gaussian elimination would require $O(N^3)$ operations, the same as for a dense matrix, because Gaussian elimination does not exploit the zero structure without modification.

The modified procedure that does exploit the structure - a forward elimination sweep that removes the sub-diagonal entries one by one, followed by back-substitution to recover the solution - runs in $O(N)$ operations. This is the Thomas algorithm, named after Llewellyn Hilleth Thomas, who described it in an unpublished 1949 Columbia University report on quantum-mechanical calculations. The method is also called the Tridiagonal Matrix Algorithm, or TDMA. It is a special case of LU factorisation, traced to Doolittle (1878) and Banachiewicz (1938), applied to a tridiagonal matrix: the L and U factors are bidiagonal, so both the factorisation and the solve reduce from $O(N^3)$ to $O(N)$. For diagonally dominant tridiagonal systems - which arise from parabolic PDE discretisations with appropriate boundary conditions - the algorithm is also numerically stable without pivoting, as shown by Isaacson and Keller (1966).

Preserving the tridiagonal structure throughout a computation is therefore not a matter of convenience. It is the condition under which the $O(N)$ cost per solve is maintained, so that refining the spatial grid by a factor of two doubles the work rather than multiplying it by eight. Any modification to the system - including the addition of penalty terms at each iteration - must keep the matrix tridiagonal for this property to hold.

Below we showed the classic algorithm model of TDMA and the specific one we introduced in Methodology with citation to Christara and Wu's work.

Algorithm 1 Thomas Algorithm (TDMA) for tridiagonal systems

```
0: Input: sub-diagonal  $a$ , main diagonal  $b$ , super-diagonal  $c$ , right-hand side  $d$ ,  
   size  $n$   
0: Output: solution vector  $x$   
   {Forward sweep}  
0: for  $i = 2, \dots, n$  do  
0:    $w \leftarrow a_i / b_{i-1}$   
0:    $b_i \leftarrow b_i - w \cdot c_{i-1}$   
0:    $d_i \leftarrow d_i - w \cdot d_{i-1}$   
0: end for  
   {Back substitution}  
0:  $x_n \leftarrow d_n / b_n$   
0: for  $i = n - 1, \dots, 1$  do  
0:    $x_i \leftarrow (d_i - c_i \cdot x_{i+1}) / b_i$   
0: end for
```

2.8.1 Penalty Iteration Algorithm

Putting everything together, the algorithm at each time step $n \rightarrow n + 1$ proceeds as follows. One starts with the initial guess $\mathbf{v}^{(0)} = \mathbf{v}^n$ and repeats:

1. From the current $\mathbf{v}^{(k)}$ determine the active regime at each node and assemble the penalty matrix $P^{(k)}$.
2. Solve the linear system

$$\left(I - \frac{\Delta\tau}{2} (\mathcal{L} + P^{(k)}) \right) \mathbf{v}^{(k+1)} = \left(I + \frac{\Delta\tau}{2} \mathcal{L} \right) \mathbf{v}^n + \Delta\tau \mathbf{g}^n.$$

3. Check both stopping criteria. If both are satisfied, accept $\mathbf{v}^{n+1} = \mathbf{v}^{(k+1)}$ and advance to the next time step; otherwise increment k and repeat.

In practice the method converges in very few iterations — typically one or two per time step on sufficiently fine grids — which is consistent with the iteration counts reported in the convergence tables presented later in this work.

Chapter 3

Problem formulation

3.1 PDE from Christara and Wu's paper

As we mentioned before, the main problem of dissertation comes from this article [2]. Control problems, which they are solved in the maximization from was

$$V_\tau = \sup_{\mu \in \{0,1\}} \sup_{Q \in \hat{Q}} \{a(S, \tau, Q)V_{SS} + b(S, \tau, Q)V_S + c(S, \tau, Q)V + d(V, \mu)\},$$

where Q are the control variables which can be 1 or more of $\{i = 1, 2, 3\}$, and μ is another control variable for only when American exercise rights are considered. So $d(V, \mu)$ is defined by

$$d(V, \mu) = \begin{cases} \mu(V^* - V)/\epsilon & \text{for American options} \\ 0 & \text{for European options,} \end{cases}$$

while the minimization form replaces the second sup in the above equation with an inf. Clearly, when European exercise rights are considered, the first sup disappears. Again for convenience, we define an operator $\mathcal{J}$ similar to $\mathcal{L}$

$$\mathcal{J}(Q)V \equiv a(S, \tau, Q)V_{SS} + b(S, \tau, Q)V_S + c(S, \tau, Q)V.$$

The equation of this work based on "Stock borrowing fees problem". Next, we give a brief description of our specific problem.

3.2 Main HJB equation

We consider a European straddle option written on an underlying asset with price S , strike price K , and maturity T . The asset price is assumed to follow a geometric Brownian motion with constant volatility $\sigma > 0$ under a risk-neutral measure. Unlike the classical Black–Scholes framework, we consider a market with frictions in which the borrowing rate r_b and the lending rate r_l are not equal, and a stock borrowing fee at rate r_f is charged to investors holding short positions. Throughout this paper we assume $r_b > r_l > r_f \geq 0$. The option value $V(S, t)$ depends on the current asset price $S \in (0, S_{\max})$ and time $t \in (0, T)$, where $S_{\max}$ is a sufficiently large truncation of the asset price domain.

The price of the European straddle option with stock borrowing fees can be obtained by solving an optimal control problem. Following Forsyth and Labahn [4] and Christara and Wu [2], the problem can be formulated as a discrete optimal control:

$$V_\tau = \sup_Q \left\{ \frac{\sigma^2 S^2}{2} V_{SS} + q_3 q_1 (SV_S - V) + (1 - q_3) ((r_l - r_f) SV_S - q_2 V) \right\}$$

with control set $Q = (q_1, q_2, q_3)$, where $q_1 \in \{r_l, r_b\}$ selects the interest rate for the fee term, $q_2 \in \{r_l, r_b\}$ selects the rate for the no-fee term, and $q_3 \in \{0, 1\}$ acts as a binary switch for the stock borrowing fee. For the long position, the sup is replaced by inf.

Applying the time reversal $\tau = T - t$ so that $\tau = 0$ corresponds to maturity, the terminal condition becomes the initial condition $V(S, 0) = |S - K|$. The corresponding nonlinear HJB PDE for the short position is:

$$\begin{cases} V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_l (SV_S - V) + \max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} \\ V(S, 0) = \max(S - K, K - S), \quad V(0, \tau) = K, \quad V(S_{\max}, \tau) = S_{\max} - K \end{cases}$$

These are precisely equation (9) from Christara and Wu [2], which we take as the starting point for our numerical treatment. We work directly in the original asset price coordinates S without applying a logarithmic transformation. The nonlinear max term reflect the optimal hedging decision at each point. For the short position let $S_1 V = (r_b - r_l)(SV_S - V)$ and $S_2 V = -r_f SV_S$, The nonlinear term is then:

$$\alpha_{\text{short}} = \begin{cases} 0 & S_1 V \leq 0 \text{ and } S_2 V \leq 0 \\ S_1 V & S_1 V > 0 \text{ and } S_1 V > S_2 V \\ S_2 V & S_2 V > 0 \text{ and } S_1 V \leq S_2 V \end{cases}$$

The short position uses r_l as the base discount rate while the long position uses r_b , which leads to different option values as shown in Section 5.

Since Christara and Wu's [2] work focused on key parameters, the 2D graphs

and tables were only for long positions, we first had to thoroughly study *Equation 10* from the same article. First, we had to determine whether we had used all the algorithms correctly and followed all the methods in the correct order, and then compare the results of the long position with an existing source. Second, since we had more than one article for comparison, it would be better to use a second one as well — my supervisor’s work, which followed the same scheme but used a different discretization method.

By the way, here equation (10)

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_l(SV_S - V) + \min\{(r_l - r_b)(SV_S - V), -(r_b - r_l + r_f)SV_S, 0\}$$

for the long position.

Furthermore, below are clear differences between tridiagonal matrices arising from the discretization:

$$P_{i,:} = \begin{cases} 0 & \text{if } [P_1 v^j]_i \geq 0 \text{ and } [P_2 v^j]_i \geq 0 \\ [P_1]_{i,:} & \text{if } [P_1 v^j]_i < 0 \text{ and } [P_1 v^j]_i < [P_2 v^j]_i \\ [P_2]_{i,:} & \text{if } [P_2 v^j]_i < 0 \text{ and } [P_1 v^j]_i \geq [P_2 v^j]_i. \end{cases}$$

for long position too. For short position, you can find same in the Methodology and Background section.

Chapter 4

Methodology

In this section we describe the complete numerical framework used to solve equations (9).

4.1 Spatial Grid

The asset price domain is discretized using $N + 1$ nodes. Since the straddle payoff $V(S, 0) = |S - K|$ has a kink at $S = K$, placing a grid node exactly at the strike price is essential for second-order convergence. On a uniform grid this non-smoothness degrades the convergence order. We use the nonuniform mapping from Clarke and Parrott [3] and Christara and Wu [2]:

$$S_i \equiv w(x_i) = \left(1 + \frac{\sinh\left(\beta\left(\frac{x_i}{x_N} - \alpha\right)\right)}{\sinh(\beta\alpha)} \right) K, \quad i = 0, 1, \dots, N$$

where $x_i = i/N$, $\alpha = 0.37$, and β is found by solving $(1 + \sinh(\beta(1 - \alpha))/\sinh(\beta\alpha))K = S_{\max}$ numerically. Dirichlet conditions $V_L = K$ and $V_R = S_{\max}$ are imposed at the boundaries. The interior unknowns are $\mathbf{u} = (u_1, \dots, u_M)^T \in \mathbb{R}^M$, $M = N - 1$.

4.2 Finite Difference Matrices

All these computations follows standard finite difference analysis (see, e.g., [6]) adapted to the nonuniform grid setting. For each interior node S_i with spacings $h_-^i = S_i - S_{i-1}$ and $h_+^i = S_{i+1} - S_i$, second-order finite difference approximations give:

$$\left. \frac{\partial V}{\partial S} \right|_{S_i} \approx \alpha_1^i V_{i-1} + \beta_1^i V_i + \gamma_1^i V_{i+1}, \quad \left. \frac{\partial^2 V}{\partial S^2} \right|_{S_i} \approx \alpha_2^i V_{i-1} + \beta_2^i V_i + \gamma_2^i V_{i+1}$$

Expand V_{i+1} and V_{i-1} by Taylor's theorem around S_i :

$$\begin{aligned} V_{i+1} &= V_i + h_+^i V_i' + \frac{(h_+^i)^2}{2} V_i'' + \frac{(h_+^i)^3}{6} V_i''' \\ V_{i-1} &= V_i - h_-^i V_i' + \frac{(h_-^i)^2}{2} V_i'' - \frac{(h_-^i)^3}{6} V_i''' \end{aligned}$$

where $V_i' = \partial V / \partial S|_{S_i}$, etc. For simplicity, the higher-order terms $O((h_+^i)^4)$, $O((h_-^i)^4)$ are not written explicitly.

So, by substituting and collecting by powers of V_i, V_i', V_i'' we will get:

$$\alpha_1^i V_{i-1} + \beta_1^i V_i + \gamma_1^i V_{i+1} = \underbrace{(\alpha_1^i + \beta_1^i + \gamma_1^i)}_{=0} V_i + \underbrace{(-h_-^i \alpha_1^i + h_+^i \gamma_1^i)}_{=1} V_i' + \underbrace{\frac{(h_-^i)^2 \alpha_1^i + (h_+^i)^2 \gamma_1^i}{2}}_{=0} V_i''$$

Here by following the methods from Christara and Wu's article, for second-order accuracy we require:

$$\begin{cases} \alpha_1^i + \beta_1^i + \gamma_1^i = 0 \\ -h_-^i \alpha_1^i + h_+^i \gamma_1^i = 1 \\ (h_-^i)^2 \alpha_1^i + (h_+^i)^2 \gamma_1^i = 0 \end{cases}$$

From the third equation: $\alpha_1^i = -\frac{(h_+^i)^2}{(h_-^i)^2} \gamma_1^i$.

Substituting into the second equation:

$$\frac{(h_+^i)^2}{h_-^i} \gamma_1^i + h_+^i \gamma_1^i = 1 \implies \gamma_1^i \cdot h_+^i \left(\frac{h_+^i}{h_-^i} + 1 \right) = 1 \implies \gamma_1^i = \frac{h_-^i}{h_+^i (h_-^i + h_+^i)}$$

Then:

$$\alpha_1^i = -\frac{(h_+^i)^2}{(h_-^i)^2} \cdot \frac{h_-^i}{h_+^i (h_-^i + h_+^i)} = \frac{-h_+^i}{h_-^i (h_-^i + h_+^i)}$$

And from the first equation:

$$\beta_1^i = -\alpha_1^i - \gamma_1^i = \frac{h_+^i}{h_-^i (h_-^i + h_+^i)} - \frac{h_-^i}{h_+^i (h_-^i + h_+^i)} = \frac{(h_+^i)^2 - (h_-^i)^2}{h_-^i h_+^i (h_-^i + h_+^i)} = \frac{h_+^i - h_-^i}{h_-^i h_+^i}$$

By exact same steps we find out all coefficients for second order derivative too, where:

$$\left. \frac{\partial^2 V}{\partial S^2} \right|_{S_i} \approx \alpha_2^i V_{i-1} + \beta_2^i V_i + \gamma_2^i V_{i+1}$$

Substituting the same Taylor expansions and collecting:

$$\alpha_2^i V_{i-1} + \beta_2^i V_i + \gamma_2^i V_{i+1} = \underbrace{(\alpha_2^i + \beta_2^i + \gamma_2^i)}_{=0} V_i + \underbrace{(-h_-^i \alpha_2^i + h_+^i \gamma_2^i)}_{=0} V_i' +$$

$$+ \underbrace{\frac{(h_-^i)^2 \alpha_2^i + (h_+^i)^2 \gamma_2^i}{2}}_{=1} V_i'' + O(h^2)$$

After computations system for Second Derivative is look like:

$$\begin{cases} \alpha_2^i + \beta_2^i + \gamma_2^i = 0 \\ -h_-^i \alpha_2^i + h_+^i \gamma_2^i = 0 \\ (h_-^i)^2 \alpha_2^i + (h_+^i)^2 \gamma_2^i = 2 \end{cases}$$

From the second equation: $\alpha_2^i = \frac{h_+^i}{h_-^i} \gamma_2^i$.

Substituting into the third:

$$(h_-^i)^2 \cdot \frac{h_+^i}{h_-^i} \gamma_2^i + (h_+^i)^2 \gamma_2^i = 2 \implies h_+^i \gamma_2^i (h_-^i + h_+^i) = 2 \implies \gamma_2^i = \frac{2}{h_+^i (h_-^i + h_+^i)}$$

Then:

$$\alpha_2^i = \frac{h_+^i}{h_-^i} \cdot \frac{2}{h_+^i (h_-^i + h_+^i)} = \frac{2}{h_-^i (h_-^i + h_+^i)}$$

And:

$$\beta_2^i = -\alpha_2^i - \gamma_2^i = -\frac{2}{h_-^i (h_-^i + h_+^i)} - \frac{2}{h_+^i (h_-^i + h_+^i)} = \frac{-2(h_-^i + h_+^i)}{h_-^i h_+^i (h_-^i + h_+^i)} = \frac{-2}{h_-^i h_+^i}$$

At the end we just need to group all coefficients of tridiagonal matrices, which are approximately define First and Second Derivatives:

$$\begin{aligned} \alpha_1^i &= \frac{-h_+^i}{h_-^i (h_-^i + h_+^i)}, & \beta_1^i &= \frac{h_+^i - h_-^i}{h_-^i h_+^i}, & \gamma_1^i &= \frac{h_-^i}{h_+^i (h_-^i + h_+^i)} \\ \alpha_2^i &= \frac{2}{h_-^i (h_-^i + h_+^i)}, & \beta_2^i &= \frac{-2}{h_-^i h_+^i}, & \gamma_2^i &= \frac{2}{h_+^i (h_-^i + h_+^i)} \end{aligned}$$

4.3 Derivative Matrices

Coefficients, which written above, define two tridiagonal matrices $T_1, T_2 \in \mathbb{R}^{M \times M}$ with boundary vectors $\mathbf{b}_{c1}$, $\mathbf{b}_{c2}$, and diagonal matrices $D_S = \text{diag}(S_1, \dots, S_M)$, $D_{S^2} = \text{diag}(S_1^2, \dots, S_M^2)$. Below you can see how these matrices looks like:

The diagonal matrix $D_S \in \mathbb{R}^{M \times M}$ is:

$$D_S = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_M \end{pmatrix}$$

The diagonal matrix $D_{S^2} \in \mathbb{R}^{M \times M}$ is:

$$D_{S^2} = \begin{pmatrix} S_1^2 & & & \\ & S_2^2 & & \\ & & \ddots & \\ & & & S_M^2 \end{pmatrix}$$

The tridiagonal matrix $T_1 \in \mathbb{R}^{M \times M}$ is:

$$T_1 = \begin{pmatrix} \beta_1^1 & \gamma_1^1 & & & \\ \alpha_1^2 & \beta_1^2 & \gamma_1^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_1^{M-1} & \beta_1^{M-1} & \gamma_1^{M-1} \\ & & & \alpha_1^M & \beta_1^M \end{pmatrix}$$

with boundary correction vectors:

$$[\mathbf{b}_{c1}]_i = \begin{cases} \alpha_1^1 V_L & i = 1 \\ 0 & 1 < i < M \\ \gamma_1^M V_R & i = M \end{cases}$$

The full first derivative approximation is then:

$$\boxed{V_S \approx T_1 u + \mathbf{b}_{c1}}$$

Of course for second order derivative analogously, $T_2 \in \mathbb{R}^{M \times M}$:

$$T_2 = \begin{pmatrix} \beta_2^1 & \gamma_2^1 & & & \\ \alpha_2^2 & \beta_2^2 & \gamma_2^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_2^{M-1} & \beta_2^{M-1} & \gamma_2^{M-1} \\ & & & \alpha_2^M & \beta_2^M \end{pmatrix}$$

with boundary vector $\mathbf{b}_{c2}$ defined with same path, and the second derivative is:

$$\boxed{V_{SS} \approx T_2 \mathbf{u} + \mathbf{b}_{c2}}$$

Boundary correction vectors $(\mathbf{b}_{c1}, \mathbf{b}_{c2})$ are found because, in the given parameters from Christara and Wu's article, we know values of $V_0 = V_L = K$ and $V_{M+1} = V_R = S_{\max}$.

4.4 Base Operators

First of all, we had to define base operator at each interior node S_i and substitute the finite difference approximations:

$$\frac{du_i}{d\tau} = \frac{\sigma^2 S_i^2}{2} V_{SS}|_{S_i} + r_l S_i V_S|_{S_i} - r_l u_i$$

$$\begin{aligned} \frac{du_i}{d\tau} &= \frac{\sigma^2 S_i^2}{2} (\alpha_2^i u_{i-1} + \beta_2^i u_i + \gamma_2^i u_{i+1} + [bc_2]_i) + r_l S_i (\alpha_1^i u_{i-1} + \beta_1^i u_i + \gamma_1^i u_{i+1} + [bc_1]_i) - r_l u_i \\ &= \left(\frac{\sigma^2 S_i^2}{2} \alpha_2^i + r_l S_i \alpha_1^i \right) u_{i-1} + \left(\frac{\sigma^2 S_i^2}{2} \beta_2^i + r_l S_i \beta_1^i - r_l \right) u_i + \left(\frac{\sigma^2 S_i^2}{2} \gamma_2^i + r_l S_i \gamma_1^i \right) u_{i+1} \\ &\quad + \frac{\sigma^2 S_i^2}{2} [bc_2]_i + r_l S_i [bc_1]_i \end{aligned}$$

Define the coefficients for each row i :

$$\begin{aligned} a_i^A &= \frac{\sigma^2 S_i^2}{2} \alpha_2^i + r_l S_i \alpha_1^i \quad (\text{lower diagonal}) \\ b_i^A &= \frac{\sigma^2 S_i^2}{2} \beta_2^i + r_l S_i \beta_1^i - r_l \quad (\text{main diagonal}) \\ c_i^A &= \frac{\sigma^2 S_i^2}{2} \gamma_2^i + r_l S_i \gamma_1^i \quad (\text{upper diagonal}) \end{aligned}$$

Finally, assembling over all interior nodes gives the tridiagonal matrix form:

$$A_{\text{short}} = \frac{\sigma^2}{2} D_{S^2} T_2 + r_l D_S T_1 - r_l I, \quad \mathbf{b}_A^{\text{short}} = \frac{\sigma^2}{2} (S_i^2) \cdot \mathbf{b}_{c2} + r_l S_i \cdot \mathbf{b}_{c1}$$

4.5 Penalty Matrices

Following Christara and Wu [2], the nonlinear terms are discretized using tridiagonal penalty matrices for the short position:

$$\begin{aligned} P_1^{\text{short}} &= (r_b - r_l)(D_S T_1 - I), \quad P_2^{\text{short}} = -r_f D_S T_1 \\ [P^{\text{short}}(\mathbf{u})]_{i,:} &= \begin{cases} \mathbf{0} & [P_1 \mathbf{u}]_i \leq 0 \text{ and } [P_2 \mathbf{u}]_i \leq 0 \\ [P_1]_{i,:} & [P_1 \mathbf{u}]_i > 0 \text{ and } [P_1 \mathbf{u}]_i > [P_2 \mathbf{u}]_i \\ [P_2]_{i,:} & [P_2 \mathbf{u}]_i > 0 \text{ and } [P_1 \mathbf{u}]_i \leq [P_2 \mathbf{u}]_i \end{cases} \end{aligned}$$

Row i of $P_1 = (r_b - r_l)(D_S T_1 - I)$:

$$a_i^{(P_1)} = (r_b - r_l) S_i \alpha_1^i, \quad b_i^{(P_1)} = (r_b - r_l) (S_i \beta_1^i - 1), \quad c_i^{(P_1)} = (r_b - r_l) S_i \gamma_1^i$$

Row i of $P_2 = -r_f D_S T_1$:

$$a_i^{(P_2)} = -r_f S_i \alpha_1^i, \quad b_i^{(P_2)} = -r_f S_i \beta_1^i, \quad c_i^{(P_2)} = -r_f S_i \gamma_1^i$$

The combined row (a_i, b_i, c_i) of $A + P(\mathbf{u})$ depends on which penalty branch is active:

Case 0 — no penalty:

$$a_i = \frac{\sigma^2 S_i^2}{2} \alpha_2^i + r_l S_i \alpha_1^i, \quad b_i = \frac{\sigma^2 S_i^2}{2} \beta_2^i + r_l S_i \beta_1^i - r_l,$$

$$c_i = \frac{\sigma^2 S_i^2}{2} \gamma_2^i + r_l S_i \gamma_1^i$$

Case 1 — P_1 active ($r_l \rightarrow r_b$ everywhere):

$$a_i = \frac{\sigma^2 S_i^2}{2} \alpha_2^i + r_b S_i \alpha_1^i, \quad b_i = \frac{\sigma^2 S_i^2}{2} \beta_2^i + r_b S_i \beta_1^i - r_b,$$

$$c_i = \frac{\sigma^2 S_i^2}{2} \gamma_2^i + r_b S_i \gamma_1^i$$

Case 2 — P_2 active ($r_l \rightarrow r_l - r_f$ in convection):

$$a_i = \frac{\sigma^2 S_i^2}{2} \alpha_2^i + (r_l - r_f) S_i \alpha_1^i, \quad b_i = \frac{\sigma^2 S_i^2}{2} \beta_2^i + (r_l - r_f) S_i \beta_1^i - r_l,$$

$$c_i = \frac{\sigma^2 S_i^2}{2} \gamma_2^i + (r_l - r_f) S_i \gamma_1^i$$

4.6 Time Discretization

The time discretization methods considered in our work are based on the general θ -scheme framework widely used in numerical analysis of parabolic partial differential equations. Here, the θ -scheme provides a representation of several classical time-stepping methods. In particular, the explicit Euler method corresponds to $\theta = 0$, the implicit Euler method corresponds to $\theta = 1$, and the Crank-Nicolson method is obtained for $\theta = 1/2$. By this formulation we get balance stability and accuracy properties within a single numerical framework. To keep notation compact write: $F^n = (A + P^n) \mathbf{u}^n + \mathbf{b}^n$.

Forward (Explicit) Euler ($\theta = 0$):

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta \tau} = F^n \implies \mathbf{u}^{n+1} = \mathbf{u}^n + \Delta \tau F^n$$

Backward (Implicit) Euler ($\theta = 1$):

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta \tau} = F^{n+1} \implies (I - \Delta \tau A) \mathbf{u}^{n+1} = \mathbf{u}^n + \Delta \tau \mathbf{b}^{n+1}$$

Crank-Nicolson ($\theta = 1/2$):

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta\tau} = \frac{1}{2}F^n + \frac{1}{2}F^{n+1} \implies \left(I - \frac{\Delta\tau}{2}A\right)\mathbf{u}^{n+1} = \left(I + \frac{\Delta\tau}{2}A\right)\mathbf{u}^n + \frac{\Delta\tau}{2}(\mathbf{b}^n + \mathbf{b}^{n+1})$$

General θ -scheme:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta\tau} = \theta F^{n+1} + (1 - \theta)F^n, \quad \theta \in [0, 1]$$

After spatial discretization both problems reduce to the nonlinear semi-discrete system $d\mathbf{u}/d\tau = (A + P(\mathbf{u}))\mathbf{u} + \mathbf{b}_A + \mathbf{b}_P(\mathbf{u})$. Applying the θ -scheme [2, 6] gives:

$$\underbrace{\left(I - \theta\Delta\tau(A + P^{n+1})\right)}_{\text{LHS}}\mathbf{u}^{n+1} = \underbrace{\left(I + (1 - \theta)\Delta\tau(A + P^n)\right)\mathbf{u}^n + \Delta\tau[\theta\mathbf{b}^{n+1} + (1 - \theta)\mathbf{b}^n]}_{\mathbf{g}^n}$$

The Crank–Nicolson–Rannacher procedure [9] sets:

$$\theta^n = \begin{cases} 1 & n = 1, 2 \quad (\text{fully implicit} - \text{Rannacher smoothing}) \\ \frac{1}{2} & n \geq 3 \quad (\text{Crank–Nicolson}) \end{cases}$$

The first two fully implicit steps suppress oscillations from the non-smooth payoff at $S = K$, after which Crank–Nicolson recovers second-order convergence. In other words this method is used for smoothness, because using CN only in first two steps gives jumps.

4.7 Penalty Iteration Algorithm

Since $P^{n+1} = P(\mathbf{u}^{n+1})$ depends on the unknown, the system is nonlinear at each time step and is solved by the following algorithm below, which is the finite difference adaptation of Algorithm 2 in Christara and Wu [2]:

Algorithm 2 Tridiagonal penalty iteration at step n , with θ -timestepping (adapted from [2])

```

1: Initialize  $\mathbf{u}^{n,0} = \mathbf{u}^{n-1}$ ,  $P^{n,0} = P(\mathbf{u}^{n-1})$ 
2: Compute  $\mathbf{g}^n = (I + (1 - \theta)\Delta\tau(A + P^{n,0}))\mathbf{u}^{n-1} + (1 - \theta)\Delta\tau \mathbf{b}^{n,0}$ 
3: for  $k = 1, \dots, \text{maxit}$  do
4:   Solve  $[I - \theta\Delta\tau(A + P^{n,k-1})]\mathbf{u}^{n,k} = \mathbf{g}^n + \theta\Delta\tau \mathbf{b}^{n,k-1}$ 
5:   if stopping criterion (C1) satisfied then
6:     break
7:   end if
8:   Compute  $P^{n,k} = P(\mathbf{u}^{n,k})$ 
9:   if stopping criterion (C2) satisfied then
10:    break
11:  end if
12: end for
13: Set  $\mathbf{u}^{n+1} = \mathbf{u}^{n,k} = 0$ 

```

The dual stopping criteria are:

$$\max_i \left\{ \frac{|u_i^{n,k} - u_i^{n,k-1}|}{\max(\text{scale}, |u_i^{n,k}|)} \right\} < \text{tol} \quad (\text{C1})$$

$$\max_i \left\{ \frac{|[P^{n,k-1}\mathbf{u}^{n,k} + \mathbf{b}_P^{n,k-1}]_i - [P^{n,k}\mathbf{u}^{n,k} + \mathbf{b}_P^{n,k}]_i|}{\max(\text{scale}, |[P^{n,k}\mathbf{u}^{n,k} + \mathbf{b}_P^{n,k}]_i|)} \right\} < \text{tol} \quad (\text{C2})$$

with $\text{tol} = 10^{-6}$, $\text{scale} = 1$, $\text{maxit} = 50$. Criterion (C1) from Forsyth and Labahn [4] stops when iterates converge. Sometimes the numbers look stable but the penalty matrix is still switching rows. That means the algorithm has not actually decided which branch of the *max* is active yet.

So C1 alone can stop too early and give a wrong answer. Criterion (C2) from Christara and Wu [2] stops when the penalty matrix stabilizes. System stops only when both are satisfied. Together they prevent oscillatory behavior near the strike and keep the average iterations per time step between 1.0 and 1.4. Furthermore, since $T1$, $T2$ and I are tridiagonal, all penalty matrices are tridiagonal, so $A + P(u)$ remains tridiagonal at every iteration and the linear system at each step is solved by direct Thomas (TDMA) factorization.

Thanks to algorithms from these primary sources [2, 6], coding on the MATLAB platform was comfortable and quick, which improved the speed of solving problems and identifying errors. On average, it took no more than 4–5 minutes to receive the iteration results and output.

4.8 Greek Sensitivities

In 1973, Fischer Black and Myron Scholes caused a sensation with a new equation that changed the trading of options once and for all. Now, given the necessary specific parameters, it was possible to determine the price of an option with mathematical precision. But in the same year, Robert C. Merton extended the theoretical framework with continuous-time hedging. Thanks to this, the classic Black-Scholes-Merton model naturally gave rise to the term “Greeks.” In fact, the Greeks are simply partial derivatives of the option value with respect to these parameters. But these very Greeks study the sensitivity to each parameter depending on the option model.

The introduction of Greeks also played a major role in legitimizing options markets and accelerating the growth of derivative exchanges such as the Chicago Board Options Exchange (CBOE). This marked the beginning of the model’s global spread for trading. Aside from these founders, Christara/Wu, and my supervisor Rakhymzhan Kazbek, there were many others who published and contributed fresh articles:

- Yi Cao, Xiaoquan Liu, Jia Zhai (2020)
“*Option valuation under no-arbitrage constraints with neural networks*”
The main goal is to improve the accuracy and stability of option valuation while preserving financial consistency.
- Matic, Packham, Härdle (2023)
“*Hedging cryptocurrency options*”
This work investigates Delta-, Gamma-, and Vega-hedging strategies for cryptocurrency options such as Bitcoin and Ethereum derivatives.
- William Nelson (2025/2026)
“*FX Options Greeks: Practical use assuming a vanilla surface*”
The study explains how Greeks are used in pricing, hedging, and managing risk for vanilla FX options.

In our case, the Greeks are obtained by post-processing the computed solution $\mathbf{u}^n$ at the final time level $\tau = T$ (corresponding to $t = 0$) using finite difference approximations of the spatial and temporal derivatives of V . We use the same nonuniform finite difference matrices T_1 and T_2 that were built for the spatial discretization, which ensures consistency and preserves the second-order accuracy of the underlying scheme.

Delta defines the rate of variation of the option price in relation to the price of the base asset:

$$\Delta_i = \left. \frac{\partial V}{\partial S} \right|_{S_i} \approx (T_1 \mathbf{u} + \mathbf{b}_{c1})_i, \quad i = 1, \dots, M$$

Gamma defines the rate of change of Delta, representing the convexity of the

option price surface:

$$\Gamma_i = \frac{\partial^2 V}{\partial S^2} \Big|_{S_i} \approx (T_2 \mathbf{u} + \mathbf{b}_{c2})_i, \quad i = 1, \dots, M$$

Theta quantifies the rate of change in the option price relative to time, calculated using a backward finite difference between the two most recent time intervals:

$$\Theta_i = -\frac{\partial V}{\partial \tau} \Big|_{S_i} \approx -\frac{V_i^{N_t} - V_i^{N_t-1}}{\Delta \tau}$$

The minus sign converts from reversed time τ back to calendar time t . The convergence of the Greeks as the grid is refined will be demonstrated numerically in Section 5.

As a supplement, we might consider other Greeks. For example, **Vega** defines the sensitivity of the option price with respect to volatility:

$$\text{Vega}_i = \frac{\partial V}{\partial \sigma} \Big|_{S_i}$$

Rho defines the sensitivity of the option price with respect to the interest rate:

$$\rho_i = \frac{\partial V}{\partial r} \Big|_{S_i}$$

Chapter 5

Numerical Results

5.1 Parameters

All of the numerical results to the SBF problem for the European straddle option in the short position are shown in this section. All computations were performed in MATLAB. The model parameters for the experiments are given in Table 5.1.

Table 5.1: Model parameters used in all numerical experiments

Parameter	Description	Value
T	Maturity	1.0
K	Strike price	100
σ	Volatility	0.3
r_b	Borrowing rate	0.05
r_l	Lending rate	0.03
r_f	Stock borrowing fee	0.004
$S_{\max}$	Domain truncation	1000

5.1.1 Short Position

The terms *long* and *short* originated long before the development of modern financial markets. And they were initially used by merchants to describe different types of trading obligations.

A *long position* means that an investor owns an asset and expects its price to increase.

Example:

- Buy a stock for \$100.
- Sell it later for \$120.
- Profit = \$20.

Therefore,

$$\text{Profit}_{\text{long}} = S_T - S_0.$$

If the asset price rises, the holder of a long position benefits.

A *short position* means that an investor first sells a borrowed asset and later repurchases it.

Example:

- Borrow a stock from a broker.
- Sell it for \$100.
- The price falls to \$80.
- Buy it back for \$80.
- Return the stock to the broker.

The profit is

$$100 - 80 = 20.$$

For a short position,

$$\text{Profit}_{\text{short}} = S_0 - S_T.$$

If the asset price decreases, the holder of a short position benefits.

Why “Long” and “Short”? Several historical explanations have been proposed for these terms. The most commonly accepted interpretation is the following:

- A *long position* requires ownership of the asset and can typically be maintained for an extended period of time.
- A *short position* involves borrowing the asset and must eventually be closed by returning the borrowed security, making it a more temporary or “shorter” commitment.

By the nineteenth century, the expressions *long stock* and *short stock* were already widely used in financial literature with essentially the same meaning as today.

The Christara and Wu’s [2] article presented more than 5 equations related to European and American options. It even demonstrated various payoff strategies, such as the straddle and the butterfly. Throughout, they used penalty or penalty-like methods and refined them with additional conditions and solutions, but however, all the tables, graphs, and solutions for these equations were specifically addressed to long positions. That is, our goal was to demonstrate the same results for short

positions—more precisely, only for the 9th equation in the article. For the short position, Christara and Wu [2] do not provide convergence tables, so we use Table 4 of Kazbek and Abdugarimova [7] as the reference benchmark, reproduced here as Table 5.2.

Table 5.2: Short position benchmark from Kazbek and Abdugarimova [7] (their Table 4). Parameters as in Table 5.1.

Nodes	Tstep	Value	Change	Ratio	Total	Average
100	27	24.0814	—	—	32	1.23
200	52	24.1215	0.0402	—	58	1.14
400	102	24.1313	0.0098	4.11	112	1.11
800	202	24.1337	0.0024	4.04	221	1.10
1600	402	24.1343	0.0006	4.02	439	1.09
3200	802	24.1345	0.0002	3.96	894	1.12

Table 5.3 shows our short position results. The converged value 24.1345121 agrees with the reference to four decimal places and the order of convergence reaches 1.99. A small difference in iteration counts compared to the reference is due to minor differences in stopping criteria implementation and does not affect the converged price.

Table 5.3: Short position, our FDM results. Parameters as in Table 5.1.

Nodes	Tstep	Value	Change	Ratio	Total	Average	Order p
101	27	24.1148105	—	—	37	1.37	—
201	52	24.1294253	1.461e-02	—	71	1.37	—
401	102	24.1332330	3.808e-03	3.84	143	1.40	1.94
801	202	24.1342046	9.717e-04	3.92	280	1.39	1.97
1601	402	24.1344503	2.457e-04	3.96	558	1.39	1.99
3201	802	24.1345121	6.176e-05	3.98	1115	1.39	1.99

For complete confidence in the accuracy of table results, here is the solution of SBF for Short position with P2-FEM and CNR methods:

Table 5.4: Solution of SBF for Short position at $t = 0$ and $K = 100$ by P2-FEM and CNR with the following parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.3$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$

Common Information					Iterations Information	
nE	N_t	Value	Change	Ratio	Total	Average
100	27	24.1362	—	—	33	1.27
200	52	24.1348	1.411e-3	—	61	1.20
400	102	24.1346	2.025e-4	6.97	115	1.14
800	202	24.1345	8.667e-5	2.34	219	1.09
1600	402	24.1345	1.709e-5	5.07	416	1.04
3200	802	24.1345	1.930e-6	8.86	813	1.01

5.1.2 Greek Sensitivities

Here we present our numerical results based on Greek options. In options trading, the delta is used to offset risk by holding a corresponding amount of the underlying asset. Gamma, on the other hand, indicates how quickly delta changes. This determines how often the hedge needs to be adjusted. Theta shows how much value the option loses each day. Because we have a nonlinear term, the Greek parameters affect the first-order spatial derivative V differently. This means that delta changes: if delta changes, the other Greeks change too. Table 5.5 presents the calculated Greek coefficients for $K = 100$ and the strike price for a short position at all levels of detail. This might be the first time that the Greeks for this nonlinear SBF HJB model have been systematically calculated in more detail than has been studied in the literature.

Table 5.5: Short position: Greeks at $S = K = 100$ across refinement levels. Parameters as in Table 5.1.

Nodes	Tstep	Value	Delta _K	Gamma _K	Theta _K	Change	Order p
101	27	24.1148105	0.218878	0.026669	-12.0905	1.461e-02	—
201	52	24.1294253	0.219062	0.026777	-12.0263	1.461e-02	0.00
401	102	24.1332330	0.219110	0.026754	-11.9950	3.808e-03	1.94
801	202	24.1342046	0.219122	0.026744	-11.9793	9.717e-04	1.97
1601	402	24.1344503	0.219125	0.026739	-11.9715	2.457e-04	1.99
3201	802	24.1345121	0.219126	0.026736	-11.9675	6.176e-05	1.99

As the grid becomes more accurate, the Greeks smoothly converge. Delta at $K = 100$ converges to approximately 0.2191, Gamma converges to approximately 0.0267, and Theta converges to approximately -11.97. The convergence of the Greeks follows the same second-order scheme as the option price itself. In other words, this confirms that the post-processing using finite differences is consistent with the underlying numerical scheme.

Figures 5.5 and 5.4 show the 2D plots of the Greeks at $t = 0$ and the 3D surfaces

over the full (S, τ) domain for $S \in [40, 200]$. All three Greeks are smooth with no spikes or oscillations, which shows that the penalty-like method handles the nonlinearity stably. The Delta curve transitions smoothly from -1 to $+1$ around the strike, Gamma has a smooth bell shape peaked near $K = 100$, and Theta is a smooth negative curve reflecting the time decay of the option value.

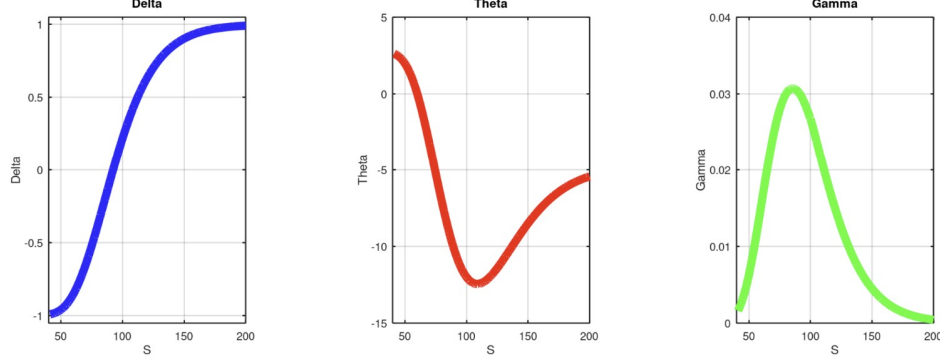


Figure 5.1: Greeks of SBF for short position at $t = 0$. Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.3$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$, $N = 400$, $N_t = 102$.

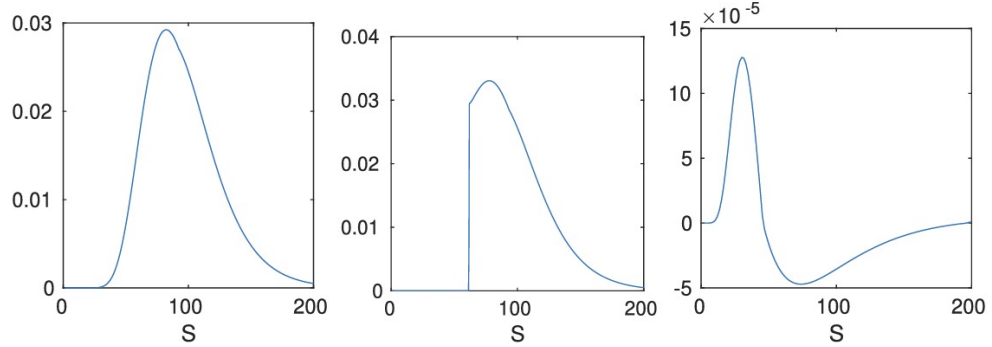


Figure 5.2: Greeks of SBF from Kazbek and Abdukarimova's paper [7] for the short position with same parameters

However, to be absolutely certain of the accuracy of the answer, we borrowed the graphs from the paper by Kazbek and Abdukarimova [7], which showed the same results but obtained using the Finite Element Matrices.

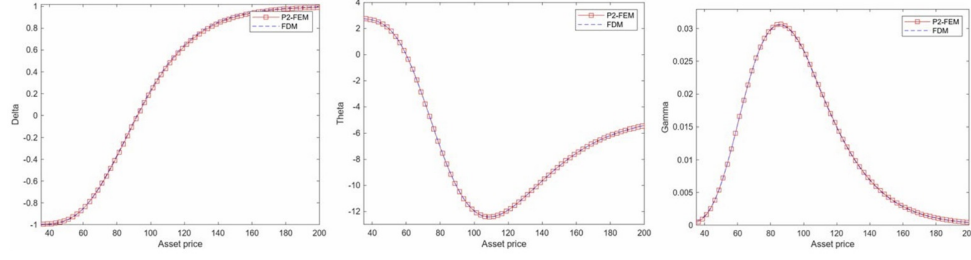


Figure 5.3: Greeks of SBF from Christara and Wu's paper [2] for the long position with same parameters

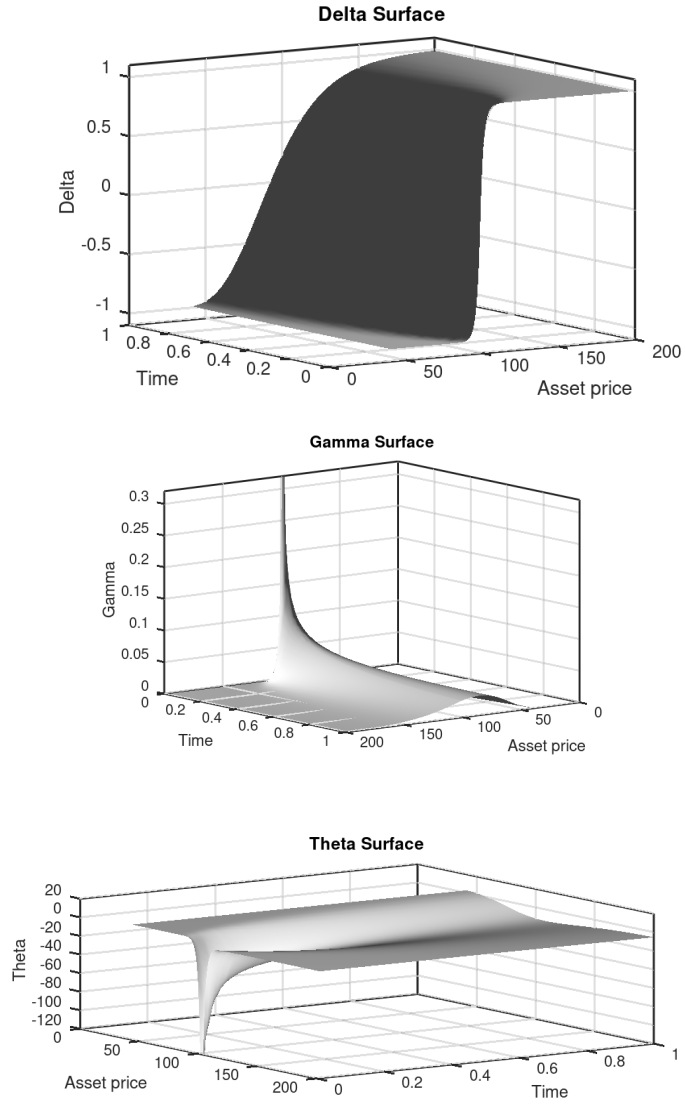


Figure 5.4: Our 3D Greek surfaces for short position. Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.3$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$, $N = 400$, $N_t = 102$.

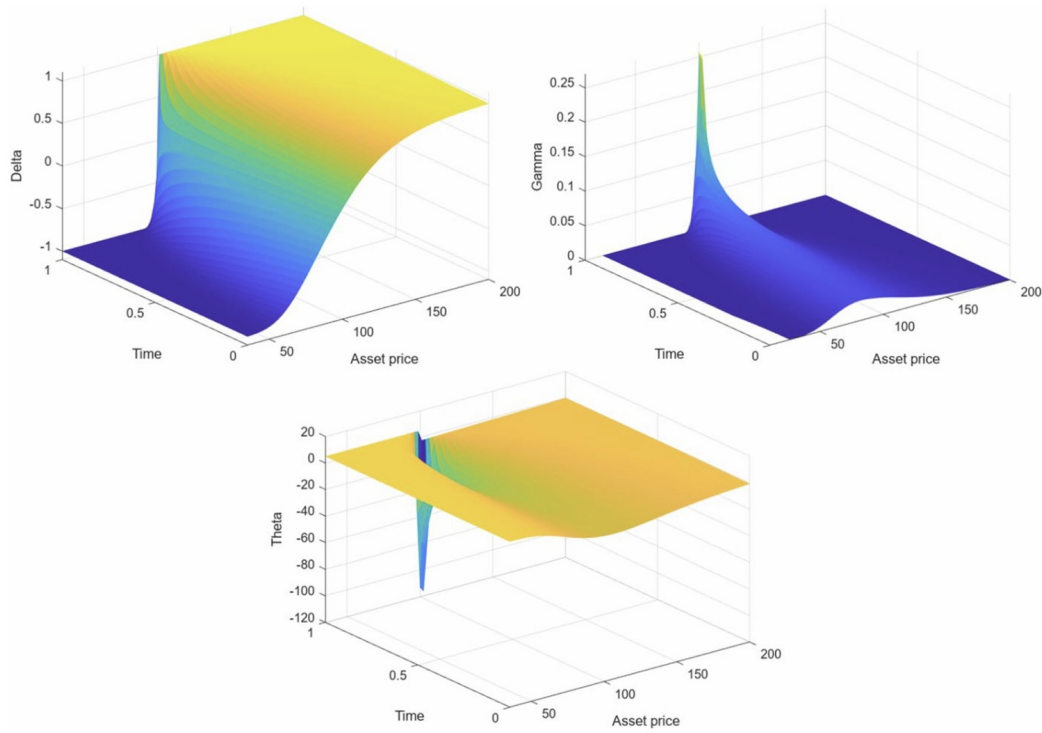


Figure 5.5: Greek surfaces of SBF for Long position by P2-FEM with the same parameters from Kazbek and Abdugarimova's paper [7]

So, by these 3D and 2D surfaces from different articles but with same problem formulation and parameters properly shows the good result of our dissertation.

Chapter 6

Final conclusions and future perspectives

6.1 Discussion

In this work, we look at the HJB problem for the European straddle with stock borrowing fees. We work with the short position and use a second-order finite difference method on a nonuniform grid. We use Crank-Nicolson-Rannacher time stepping and a penalty-like iterative solver. Our results show second-order convergence for the short position. We compare with the benchmarks of Forsyth and Labahn [4] and Christara and Wu [2]. The main result of this work is the first systematic computation of the Greeks, Delta, Gamma, and Theta, for the short position in the nonlinear SBF HJB model using finite difference methods. They demonstrate second-order convergence and remain smooth for all grid sizes, without jumps or abrupt transitions.

In these option problems, Kazbek and Abdugarimova [7] used the FEM for the short position, while Christara and Wu [2] considered the long position. However, no full finite-difference computation of the Greek coefficients for the short position had previously been reported. Our results fill this small gap and show that exact Greek coefficients can be obtained in the same way, presenting the exact results in the form of graphs 5.5–5.4 and table 5.5.

6.2 Conclusions

We started with the standard European option and built the results step by step using the pricing framework [1, 6], then adding the penalty-like iteration, and then calculated the Greeks. After that, the penalty matrices and dual stopping criteria from Christara and Wu [2], second-order convergence was recovered for the short position and the values matched the benchmarks of [2, 7].

This paper shows that the Greeks, when computed by post-processing, using finite differences, also converge at second order. Table 5.5 shows this result. The

results in Figures 5.5–5.4 are smooth and do not have spikes, which shows the method works well. The literature already proves that the HJB problem has a unique solution and that the penalty method converges to the viscosity solution. For more details, see the appendix of Christara and Wu [2].

6.3 Future work

While doing this work, we became familiar with new methods, types of PDE equations, and solution techniques thanks to Ali Hirs’s book [1, 6] and the collection of solutions by Christara and Wu [2]. Naturally, we didn’t start right away with the 9th equation from this article; we began with the first one and worked my way up to the equation presented here.

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \text{nonlinear terms},$$

$$\begin{aligned} \text{nonlinear terms} &= \max\{\mathcal{L}_1 V, \mathcal{L}_2 V, 0\} + \rho \max\{V^* - V, 0\} \\ \rho &= \begin{cases} 1/\epsilon & \text{for American options, } \epsilon \text{ is a small positive constant} \\ 0 & \text{for European options} \end{cases} \end{aligned}$$

From above, the very nearest problem could be adding another nonlinear term with $\rho \geq 0$. Only by this formulation we could automatically change the European equation to the American one. And there are still 5+ equations there, not only for European options but also for American ones. Expanding on penalty-like methods and familiarizing myself with penalty matrices for American options—there is truly room to grow. Because, as we’ve noted many times above, the solutions in these articles were published only for long positions. This alone opens up opportunities to take small steps toward academic work. For example, there is 2 next problems from Christara and Wu’s paper.

Stock borrowing fees problem with American exercise rights:

$$\begin{aligned} V_\tau &= \frac{\sigma^2 S^2}{2} V_{SS} + r_l(SV_S - V) + \max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} + \\ &\quad + \rho \max\{V^* - V, 0\} \end{aligned}$$

for the short position, and

$$\begin{aligned} V_\tau &= \frac{\sigma^2 S^2}{2} V_{SS} + r_l(SV_S - V) + \min\{(r_l - r_b)(SV_S - V), -(r_b - r_l + r_f)SV_S, 0\} + \\ &\quad + \rho \max\{V^* - V, 0\} \end{aligned}$$

for the long position (which already has the solution in paper). It’s also worth noting that in this paper we examined the straddle payoff, whereas in the original source the authors also presented the butterfly spread payoff method. This is the

equation (15) and table of solutions from same article:

$$V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \min\{\kappa S^2 V_{SS}, -\kappa S^2 V_{SS}\}.$$

Table 6.1: European transaction cost model with butterfly spread payoff and constant timesteps; value computed at K . Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $a = 10$, $\sigma = 0.65$, $r = 0.05$, $\kappa = 0.1$.

Nodes	Tstep	Value	Change	Rate	Total	Average
101	102	0.126405	—	—	121	1.19
201	202	0.125742	-6.63e-04	—	236	1.17
401	402	0.125485	-2.57e-04	1.37	474	1.18
801	802	0.125361	-1.24e-04	1.05	936	1.17
1601	1602	0.125323	-3.83e-05	1.70	1879	1.17
3201	3202	0.125311	-1.20e-05	1.68	3736	1.17

Furthermore, we should use the works of my current academic advisor, Rakhymzhan Kazbek, as authoritative reference sources for comparing results. That will be a nice start for future research work from this point.

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Numerical Methods for HJB Equations in Option Pricing with Greek Sensitivities

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Abstract

Usually, optimal control problems in financial mathematics are solved by maximizing profit and using the Hamilton-Jacobi-Bellman equation. This leads to nonlinear PDE, the solution of which is based on numerical methods. In this paper, we use the method described in the article by Christara and Wu: second-order finite differences on a non-uniform grid. However, instead of the long position discussed there, we study the short position. Therefore, here too we use the same time step, calculated using the Crank-Nicolson-Rannacher method, and we employ an iterative solver with a penalty term. The nonuniform grid puts more nodes near the strike price. This is important for keeping second-order convergence when the straddle payoff is not smooth. Rannacher smoothing at the first time steps helps remove unwanted oscillations from the kink in the initial condition. The nonlinear penalty-like term comes from the optimal control in the stock borrowing fee model. We solve it at each time step with a Newton-type iterative algorithm that uses two stopping criteria. Our numerical tests show second-order convergence for the short position. The results agree with the benchmarks of Forsyth and Labahn and also Christara and Wu. The main result of this work is the numerical calculation of the Greeks for this nonlinear HJB model: Delta, Gamma, and Theta. As far as we know, there is no other full numerical study of the Greeks for the short position using finite differences on HJB problems. This is the main applied result of our work.

Keywords: HJB equations, stock borrowing fees, finite difference method, penalty-like iteration, option Greeks

1 Introduction

Option pricing is a key challenge in mathematical finance. The classical Black-Scholes model [1] is a well-known framework for valuing European options, but it relies on several assumptions that simplify reality, and these do not always match real market conditions. The Black-Scholes equation cannot directly account for situations where there are borrowing fees. For example, if a trader wants to open a short position early on stocks. Because we now take into account the borrowing and lending fees, our equation, of course, is no longer linear. Instead, it transforms into the Hamilton-Jacobi-Bellman (HJB) equation, which is one of the fundamental PDE problems in optimal control. Here, we can choose the best possible hedging strategy at every point in time. For a short position, the main equation includes a max term, and for a long position, a min term, respectively. From a numerical perspective, the HJB equation cannot be solved using standard linear equation solvers. And the first to study these equations using numerical methods were Forsyth and Labahn [3]. They analyzed the SBF problem step by step and formulated it as the optimal control problem we know today. To solve it, they applied the policy iteration method [12] in combination with the Crank-Nicolson time step. Their results for the European straddle option are now widely accepted as the gold standard. My goal in this article, together with my supervisor Rakhymzhan Kazbek, was to reproduce and extend their convergence tables using Greeks. In 2002, Forsyth and Vetzal [10] introduced the concept of iteration with a penalty term when working with American options. As a result, this method also proved successful and is still used by modern mathematicians today. Christara and Wu [6] improved upon earlier methods involving penalty terms [10, 11]. They found an alternative way to handle the non-linear max and min terms in the FEM equations. They used triangular penalty matrices that fit easily into the standard finite difference scheme. Furthermore, they proposed two stopping criteria that work together—one based on the solution and the other on the penalty matrix.

In this paper, we directly implement and extend the framework of Christara and Wu [6], working specifically with their PDE equation (9) for the short-term nonlinear version of the SBF problem. Paper focused on the European straddle option with stock borrowing fees. We checked our numerical results against the reference results obtained using the finite difference method by Forsyth and Laban [3], and compared them with the results of the finite element method by Kazbek and Abdukarymova [7]. While Christara and Wu [6] mainly has results for a long position with a minimum term, our goal was to add a full study of the Greeks specifically for the short position. Because of this our paper tries to fill this small but nonetheless important gap in the existing literature. Therefore, we used the same methods and algorithms for MATLAB as in the paper: a second-order finite difference method on a nonuniform grid with Crank-Nicolson-Rannacher time stepping [9]. Of course, we also

handled the nonlinear term using a penalty-type iteration with two stopping criteria. But the main small novelty was the implementation and analysis of the Greeks (Delta, Gamma, and Theta) on the short position.

Sure, we didn't immediately start studying the HJB equation and what optimal solutions we could use. To understand how the algorithm works and how the calculations were performed, we had to start with the Black-Scholes [1] equation. More specifically, we needed to discretize the linear equation and transform it into a tridiagonal form so that we could verify convergence using modern programming languages such as MATLAB. To do this, we relied on Hirsa [8]'s book, which provides a very detailed analysis of the computations and includes analytical background. Thanks to this book, it was clear how to correctly add the nonlinear part step by step and how to switch between solution methods. For example, we started with the explicit Euler method, then moved on to the implicit method, and later used the Crank-Nicolson-Rannacher method with implementing Greeks. It worked very well and allowed us to evaluate the convergence and the results.

The problem of nonlinear option pricing can be solved using other methods as well; one of them is the finite element method. Most recently, in 2024, my supervisor Kazbek, along with Erlangga, Amanbek and Wei [4], applied finite element methods to the pricing of convertible bonds using a penalty function-based approach. They demonstrated that higher-order basis functions can improve both accuracy and computational efficiency. Building on this work, Kazbek and Abdukarimova [7] extended the method specifically for the SBF HJB problem, using quadratic Lagrangian elements on a non-uniform grid in combination with the Crank-Nicolson-Rannacher time stepping [9]. Since the parameters in our article are the same as theirs, and the calculation method is also the same, their article served as another point of comparison for the results. And it worked really well, even though the spatial discretization was different—in the end, we obtained nearly identical numerical values. This allows a direct comparison of the two methodologies in terms of convergence behavior and computational cost.

Even with the progress above, numerically finding the Greeks for the SBF-HJB model, especially for short positions, is still not fully solved. The Greeks are important for hedging in practice, because they show how the option price changes when the underlying asset price or time changes. In the nonlinear HJB case, the Greeks behave more complicated than in the usual Black-Scholes model. So, to compute them accurately, we need to handle the same numerical problems as in pricing. treatment of the same numerical difficulties that arise in the pricing problem itself. The This paper aims to fill this gap by using the finite difference method described above to compute and study the Greeks numerically.

2 Model Formulation

We consider a European straddle option written on an underlying asset with price S , strike price K , and maturity T . The asset price is assumed to follow a geometric Brownian motion with constant volatility $\sigma > 0$ under a risk-neutral measure. Unlike the classical Black–Scholes framework, we consider a market with frictions in which the borrowing rate r_b and the lending rate r_l are not equal, and a stock borrowing fee at rate r_f is charged to investors holding short positions. Throughout this paper we assume $r_b > r_l > r_f \geq 0$. The option value $V(S, t)$ depends on the current asset price $S \in (0, S_{\max})$ and time $t \in (0, T)$, where $S_{\max}$ is a sufficiently large truncation of the asset price domain.

The price of the European straddle option with stock borrowing fees can be obtained by solving an optimal control problem. Following Forsyth and Labahn [3] and Christara and Wu [6], the problem can be formulated as a discrete optimal control:

$$V_\tau = \sup_Q \left\{ \frac{\sigma^2 S^2}{2} V_{SS} + q_3 q_1 (SV_S - V) + (1 - q_3) ((r_l - r_f) SV_S - q_2 V) \right\} \quad (1)$$

with control set $Q = (q_1, q_2, q_3)$, where $q_1 \in \{r_l, r_b\}$ selects the interest rate for the fee term, $q_2 \in \{r_l, r_b\}$ selects the rate for the no-fee term, and $q_3 \in \{0, 1\}$ acts as a binary switch for the stock borrowing fee. For the long position, the sup is replaced by inf.

Applying the time reversal $\tau = T - t$ so that $\tau = 0$ corresponds to maturity, the terminal condition becomes the initial condition $V(S, 0) = |S - K|$. The corresponding nonlinear HJB PDE for the **short position** is:

$$\begin{cases} V_\tau = \frac{\sigma^2 S^2}{2} V_{SS} + r_l (SV_S - V) + \max\{(r_b - r_l)(SV_S - V), -r_f SV_S, 0\} \\ V(S, 0) = \max(S - K, K - S), \quad V(0, \tau) = K, \quad V(S_{\max}, \tau) = S_{\max} - K \end{cases} \quad (2)$$

These are precisely equation (9) from Christara and Wu [6], which we take as the starting point for our numerical treatment. We work directly in the original asset price coordinates S without applying a logarithmic transformation. The nonlinear max term reflect the optimal hedging decision at each point. For the short position let $S_1 V = (r_b - r_l)(SV_S - V)$ and $S_2 V = -r_f SV_S$, The nonlinear term is then:

$$\alpha_{\text{short}} = \begin{cases} 0 & S_1 V \leq 0 \text{ and } S_2 V \leq 0 \\ S_1 V & S_1 V > 0 \text{ and } S_1 V > S_2 V \\ S_2 V & S_2 V > 0 \text{ and } S_1 V \leq S_2 V \end{cases} \quad (3)$$

The short position uses r_l as the base discount rate while the long position uses r_b , which leads to different option values as shown in Section 5.

3 Methodology

In this section we describe the complete numerical framework used to solve equations (9).

3.1 Spatial Grid

The asset price domain is discretized using $N + 1$ nodes. Since the straddle payoff $V(S, 0) = |S - K|$ has a kink at $S = K$, placing a grid node exactly at the strike price is essential for second-order convergence. We use the nonuniform mapping from Clarke and Parrott [5] and Christara and Wu [6]:

$$S_i \equiv w(x_i) = \left(1 + \frac{\sinh\left(\beta\left(\frac{x_i}{x_N} - \alpha\right)\right)}{\sinh(\beta\alpha)} \right) K, \quad i = 0, 1, \dots, N \quad (4)$$

where $x_i = i/N$, $\alpha = 0.37$, and β is found by solving $(1 + \sinh(\beta(1 - \alpha))/\sinh(\beta\alpha))K = S_{\max}$ numerically. Dirichlet conditions $V_L = K$ and $V_R = S_{\max}$ are imposed at the boundaries. The interior unknowns are $\mathbf{u} = (u_1, \dots, u_M)^T \in \mathbb{R}^M$, $M = N - 1$.

3.2 Finite Difference Matrices

For each interior node S_i with spacings $h_-^i = S_i - S_{i-1}$ and $h_+^i = S_{i+1} - S_i$, second-order finite difference approximations give:

$$\left. \frac{\partial V}{\partial S} \right|_{S_i} \approx \alpha_1^i V_{i-1} + \beta_1^i V_i + \gamma_1^i V_{i+1}, \quad \left. \frac{\partial^2 V}{\partial S^2} \right|_{S_i} \approx \alpha_2^i V_{i-1} + \beta_2^i V_i + \gamma_2^i V_{i+1} \quad (5)$$

where:

$$\alpha_1^i = \frac{-h_+^i}{h_-^i(h_-^i + h_+^i)}, \quad \beta_1^i = \frac{h_+^i - h_-^i}{h_-^i h_+^i}, \quad \gamma_1^i = \frac{h_-^i}{h_+^i(h_-^i + h_+^i)} \quad (6)$$

$$\alpha_2^i = \frac{2}{h_-^i(h_-^i + h_+^i)}, \quad \beta_2^i = \frac{-2}{h_-^i h_+^i}, \quad \gamma_2^i = \frac{2}{h_+^i(h_-^i + h_+^i)} \quad (7)$$

We note that Christara and Wu [6] use a uniform grid, so these nonuniform coefficients were derived independently and validated against their benchmark. They define two tridiagonal matrices $T_1, T_2 \in \mathbb{R}^{M \times M}$ with boundary vectors $\mathbf{b}_{c1}$, $\mathbf{b}_{c2}$, and diagonal matrices $D_S = \text{diag}(S_1, \dots, S_M)$, $D_{S^2} = \text{diag}(S_1^2, \dots, S_M^2)$.

3.3 Base Operators

Substituting the finite difference approximations into the linear parts of equation (9) and assembling over all interior nodes gives:

$$A_{\text{short}} = \frac{\sigma^2}{2} D_{S^2} T_2 + r_l D_S T_1 - r_l I, \quad \mathbf{b}_A^{\text{short}} = \frac{\sigma^2}{2} (S_i^2) \cdot \mathbf{b}_{c2} + r_l S_i \cdot \mathbf{b}_{c1} \quad (8)$$

3.4 Penalty Matrices

Following Christara and Wu [6], the nonlinear terms are discretized using tridiagonal penalty matrices for the short position:

$$P_1^{\text{short}} = (r_b - r_l)(D_S T_1 - I), \quad P_2^{\text{short}} = -r_f D_S T_1 \quad (9)$$

$$[P^{\text{short}}(\mathbf{u})]_{i,:} = \begin{cases} \mathbf{0} & [P_1 \mathbf{u}]_i \leq 0 \text{ and } [P_2 \mathbf{u}]_i \leq 0 \\ [P_1]_{i,:} & [P_1 \mathbf{u}]_i > 0 \text{ and } [P_1 \mathbf{u}]_i > [P_2 \mathbf{u}]_i \\ [P_2]_{i,:} & [P_2 \mathbf{u}]_i > 0 \text{ and } [P_1 \mathbf{u}]_i \leq [P_2 \mathbf{u}]_i \end{cases} \quad (10)$$

3.5 Time Discretization

After spatial discretization both problems reduce to the nonlinear semi-discrete system $d\mathbf{u}/d\tau = (A + P(\mathbf{u}))\mathbf{u} + \mathbf{b}_A + \mathbf{b}_P(\mathbf{u})$. Applying the θ -scheme gives:

$$\underbrace{\left(I - \theta \Delta \tau (A + P^{n+1}) \right)}_{\text{LHS}} \mathbf{u}^{n+1} = \underbrace{\left(I + (1 - \theta) \Delta \tau (A + P^n) \right) \mathbf{u}^n + \Delta \tau [\theta \mathbf{b}^{n+1} + (1 - \theta) \mathbf{b}^n]}_{\mathbf{g}^n} \quad (11)$$

The Crank–Nicolson–Rannacher procedure [9] sets:

$$\theta^n = \begin{cases} 1 & n = 1, 2 \quad (\text{fully implicit — Rannacher smoothing}) \\ \frac{1}{2} & n \geq 3 \quad (\text{Crank–Nicolson}) \end{cases} \quad (12)$$

The first two fully implicit steps suppress oscillations from the non-smooth payoff at $S = K$, after which Crank–Nicolson recovers second-order convergence.

3.6 Penalty Iteration Algorithm

Since $P^{n+1} = P(\mathbf{u}^{n+1})$ depends on the unknown, the system is nonlinear at each time step and is solved by the following algorithm, which is the finite difference adaptation of Algorithm 2 in Christara and Wu [6].

The dual stopping criteria are:

$$\max_i \left\{ \frac{|u_i^{n,k} - u_i^{n,k-1}|}{\max(\text{scale}, |u_i^{n,k}|)} \right\} < \text{tol} \quad (\text{C1})$$

$$\max_i \left\{ \frac{|[P^{n,k-1} \mathbf{u}^{n,k} + \mathbf{b}_P^{n,k-1}]_i - [P^{n,k} \mathbf{u}^{n,k} + \mathbf{b}_P^{n,k}]_i|}{\max(\text{scale}, |[P^{n,k} \mathbf{u}^{n,k} + \mathbf{b}_P^{n,k}]_i|)} \right\} < \text{tol} \quad (\text{C2})$$

with $\text{tol} = 10^{-6}$, $\text{scale} = 1$, $\text{maxit} = 50$. Criterion (C1) from Forsyth and Labahn [3] stops when iterates converge. Criterion (C2) from Christara and Wu [6] stops when the penalty matrix stabilizes. Together they prevent oscillatory behavior near the strike and keep the average iterations per time step between 1.0 and 1.4.

Algorithm 1 Tridiagonal penalty iteration at step n , with θ -timestepping (adapted from [6])

```

1: Initialize  $\mathbf{u}^{n,0} = \mathbf{u}^{n-1}$ ,  $P^{n,0} = P(\mathbf{u}^{n-1})$ 
2: Compute  $\mathbf{g}^n = (I + (1 - \theta)\Delta\tau(A + P^{n,0}))\mathbf{u}^{n-1} + (1 - \theta)\Delta\tau \mathbf{b}^{n,0}$ 
3: for  $k = 1, \dots, \text{maxit}$  do
4:   Solve  $[I - \theta\Delta\tau(A + P^{n,k-1})]\mathbf{u}^{n,k} = \mathbf{g}^n + \theta\Delta\tau \mathbf{b}^{n,k-1}$ 
5:   if stopping criterion (C1) satisfied then
6:     break
7:   end if
8:   Compute  $P^{n,k} = P(\mathbf{u}^{n,k})$ 
9:   if stopping criterion (C2) satisfied then
10:    break
11:  end if
12: end for
13: Set  $\mathbf{u}^{n+1} = \mathbf{u}^{n,k}$ 

```

3.7 Greek Sensitivities

The option Greeks are obtained by post-processing the computed solution $\mathbf{u}^n$ at the final time level $\tau = T$ (corresponding to $t = 0$) using finite difference approximations of the spatial and temporal derivatives of V . We use the same nonuniform finite difference matrices T_1 and T_2 that were built for the spatial discretization, which ensures consistency and preserves the second-order accuracy of the underlying scheme.

Delta defines the rate of variation of the option price in relation to the price of the base asset:

$$\Delta_i = \frac{\partial V}{\partial S} \Big|_{S_i} \approx (T_1 \mathbf{u} + \mathbf{b}_{c1})_i, \quad i = 1, \dots, M \quad (13)$$

Gamma defines the rate of change of Delta, representing the convexity of the option price surface:

$$\Gamma_i = \frac{\partial^2 V}{\partial S^2} \Big|_{S_i} \approx (T_2 \mathbf{u} + \mathbf{b}_{c2})_i, \quad i = 1, \dots, M \quad (14)$$

Theta quantifies the rate of change in the option price relative to time, calculated using a backward finite difference between the two most recent time intervals:

$$\Theta_i = -\frac{\partial V}{\partial \tau} \Big|_{S_i} \approx -\frac{V_i^{N_t} - V_i^{N_t-1}}{\Delta\tau} \quad (15)$$

The minus sign converts from reversed time τ back to calendar time t . The convergence of the Greeks as the grid is refined will be demonstrated numerically in Section 5.

4 Numerical Results

All of the numerical results to the SBF problem for the European straddle option in the short position are shown in this section. All computations were performed in MATLAB. The model parameters for the experiments are given in Table 1.

Table 1: Model parameters used in all numerical experiments

Parameter	Description	Value
T	Maturity	1.0
K	Strike price	100
σ	Volatility	0.3
r_b	Borrowing rate	0.05
r_l	Lending rate	0.03
r_f	Stock borrowing fee	0.004
$S_{\max}$	Domain truncation	1000

4.1 Short Position

For the short position, Christara and Wu [6] do not provide convergence tables, so we use Table 4 of Kazbek and Abdugarimova [7] as the reference benchmark, reproduced here as Table 2.

Table 2: Short position benchmark from Kazbek and Abdugarimova [7] (their Table 4). Parameters as in Table 1.

Nodes	Tstep	Value	Change	Ratio	Total	Average
100	27	24.0814	—	—	32	1.23
200	52	24.1215	0.0402	—	58	1.14
400	102	24.1313	0.0098	4.11	112	1.11
800	202	24.1337	0.0024	4.04	221	1.10
1600	402	24.1343	0.0006	4.02	439	1.09
3200	802	24.1345	0.0002	3.96	894	1.12

Table 3 shows our short position results. The converged value 24.1345121 agrees with the reference to four decimal places and the order of convergence reaches 1.99. A small difference in iteration counts compared to the reference is due to minor differences in stopping criteria implementation and does not affect the converged price.

Table 3: Short position, our FDM results. Parameters as in Table 1.

Nodes	Tstep	Value	Change	Ratio	Total	Average	Order p
101	27	24.1148105	—	—	37	1.37	—
201	52	24.1294253	1.461e-02	—	71	1.37	—
401	102	24.1332330	3.808e-03	3.84	143	1.40	1.94
801	202	24.1342046	9.717e-04	3.92	280	1.39	1.97
1601	402	24.1344503	2.457e-04	3.96	558	1.39	1.99
3201	802	24.1345121	6.176e-05	3.98	1115	1.39	1.99

4.2 Greek Sensitivities

Here we present our numerical results based on Greek options. In options trading, the delta is used to offset risk by holding a corresponding amount of the underlying asset. Gamma, on the other hand, indicates how quickly delta changes. This determines how often the hedge needs to be adjusted. Theta shows how much value the option loses each day. Because we have a nonlinear term, the Greek parameters affect the first-order spatial derivative V differently. This means that delta changes: if delta changes, the other Greeks change too. Table 4 presents the calculated Greek coefficients for $K = 100$ and the strike price for a short position at all levels of detail. This might be the first time that the Greeks for this nonlinear SBF HJB model have been systematically calculated in more detail than has been studied in the literature.

Table 4: Short position: Greeks at $S = K = 100$ across refinement levels. Parameters as in Table 1.

Nodes	Tstep	Value	Delta $_K$	Gamma $_K$	Theta $_K$	Change	Order p
101	27	24.1148105	0.218878	0.026669	-12.0905	1.461e-02	—
201	52	24.1294253	0.219062	0.026777	-12.0263	1.461e-02	0.00
401	102	24.1332330	0.219110	0.026754	-11.9950	3.808e-03	1.94
801	202	24.1342046	0.219122	0.026744	-11.9793	9.717e-04	1.97
1601	402	24.1344503	0.219125	0.026739	-11.9715	2.457e-04	1.99
3201	802	24.1345121	0.219126	0.026736	-11.9675	6.176e-05	1.99

As the grid becomes more accurate, the Greeks smoothly converge. Delta at $K = 100$ converges to approximately 0.2191, Gamma converges to approximately 0.0267, and Theta converges to approximately -11.97. The convergence of the Greeks follows the same second-order scheme as the option price itself. In other words, this confirms that the post-processing

using finite differences is consistent with the underlying numerical scheme.

Figures 1 and ?? show the 2D plots of the Greeks at $t = 0$ and the 3D surfaces over the full (S, τ) domain for $S \in [40, 200]$. All three Greeks are smooth with no spikes or oscillations, which shows that the penalty-like method handles the nonlinearity stably. The Delta curve transitions smoothly from -1 to $+1$ around the strike, Gamma has a smooth bell shape peaked near $K = 100$, and Theta is a smooth negative curve reflecting the time decay of the option value.

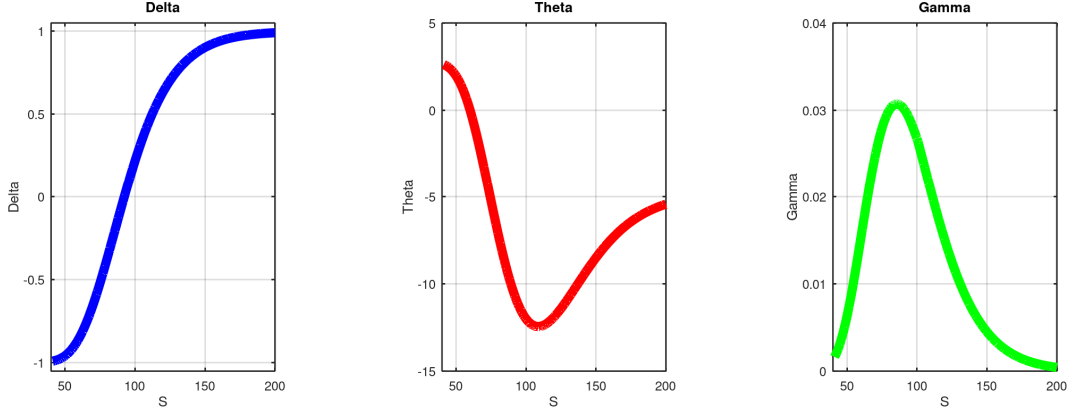


Figure 1: Greeks of SBF for short position at $t = 0$. Parameters: $T = 1$, $S_{\max} = 1000$, $K = 100$, $\sigma = 0.3$, $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$, $N = 400$, $N_t = 102$.

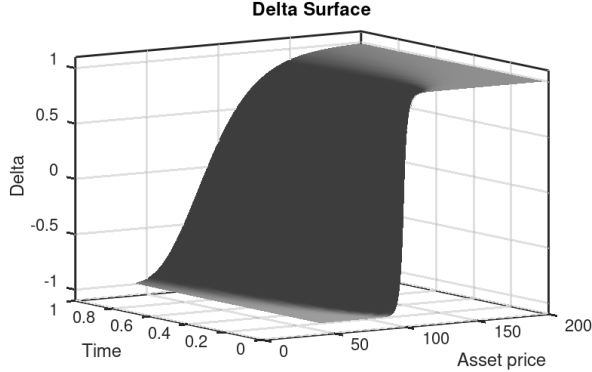


Figure 3: Enter Caption

4.3 Discussion

We started with the standard European option and built the results step by step using the pricing framework [1, 8], then adding the penalty-like iteration, and then calculated the Greeks. After that, the penalty matrices and dual stopping criteria from Christara and Wu [6], second-order convergence was recovered for the short position and the values matched the benchmarks of [6, 7].

This paper shows that the Greeks, when computed by post-processing, using finite differences, also converge at second order. Table 4 shows this result. The results in Figures 1-?? are smooth and do not have spikes, which shows the method works well. The literature already proves that the HJB problem has a unique solution and that the penalty method converges to the viscosity solution. For more details, see the appendix of Christara and Wu [6].

5 Conclusion

In this paper, we look at the HJB problem for the European straddle with stock borrowing fees. We work with the short position and use a second-order finite difference method on a nonuniform grid. We use Crank-Nicolson-Rannacher time stepping and a penalty-like iterative solver. Our results show second-order convergence for the short position. We compare with the benchmarks of Forsyth and Labahn [3] and Christara and Wu [6]. The main result of this work is the first systematic computation of the Greeks, Delta, Gamma, and Theta, for the short position in the nonlinear SBF HJB model using finite difference methods. They demonstrate second-order convergence and remain smooth for all grid sizes, without jumps or abrupt transitions.

In these option problems, Kazbek and Abdukarimova [7] used the FEM for the short

position, while Christara and Wu [6] considered the long position. However, no full finite-difference computation of the Greek coefficients for the short position had previously been reported. Our results fill this small gap and show that exact Greek coefficients can be obtained in the same way, presenting the exact results in the form of graphs [1]–?? and table [4]. We want to extend this to American-style options in the future, which will need a variational inequality and a double-penalty method.

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