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Method of generalized Laguerre polynomials in problems with a moving boundary for the generalized heat equation

THESIS

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Abstract

This thesis is about solving Heat equation. exactly. generalized heat equation. which. solved using Laguerre polynomials. The heat equation is an important partial differential equation (PDE) which describes the distribution of heat (or variation in temperature) in a given region over time It considers heat transferring in electrical contact from one side to another. Simple examples to the Generalized Heat equation are the melting of ice and the freezing of water. Solving the Heat equation, known problem in numerous industrial and technological applications. such as the manufacture of steel. ablation of heat shields. contact melting in thermal storage systems. ice accretion on aircraft. evaporation of water.

Generalized Heat equation with moving boundary have been solved by Laplace Transform. My aim is to solve the same equation by Laguerre polynomials obtained from heat polynomials.

Аннотация

Этот тезис о решении уравнения теплопроводности , а именно, обобщенного уравнения тепла, который решается с помощью полиномов Лагерра. Уравнение теплопроводности является важным уравнением в частных производных, которое описывает распределение тепла (или изменение температуры) в данном регионе во времени в электрическом контакте с одной стороны на другую. Простые примеры к обобщенном Уравнению теплопроводности - это таяние льда и замерзание воды. Решая Уравнения теплопроводности, известная проблема в многочисленных промышленных и технологических приложениях, такие как производство стали, абляция теплозащитных экранов, контактная плавка в системы накопления тепла, прирост льда на самолетах, испарение воды.

Обобщенное Уравнение теплопроводности с движущейся границей было решено переобразованием Лапласа. Моя цель состоит в том, чтобы решить то же самое уравнение полиномами Лагерра, полученными из тепловых полиномов.

Аңдатпа

Осы диссертациялық жұмыс жылу теңдеуін шешу жайлы жазылған, нақтырақ айтқанда, Лагерь полиномдарын қолдана отырып жалпыланған жылу теңдеуін шешу. Жылулық теңдеуі – белгілі бір аймақтағы уақыт бойынша жылудың (немесе температураның өзгерісінің) бір жағынан екінші жағына электрлік байланыста таралуын сипаттайтын маңызды жартылай дифференциалдық теңдеу. Жалпыланған жылу теңдеулеріне қарапайым мысалдар мұздың еруі мен суды қатыру болып табылады. Жылулық теңдеулер теңдеулерін шешу көптеген өнеркәсіптік және технологиялық қосымшаларда танымал мәселе болып табылады. атап айтқанда , болат өндірісі, жылу қалқандары абляциясы, жылу сақтау жүйелеріндегі байланыс балқыту, ұшақтардағы мұзды қалыптастыру және судың булануы сияқты.

Осыған дейін, қозғалыс шекарасы бар жылу өткізгіштің жалпыланған теңдеуі Лаплас түрлендіруімен шешілген болатын. Менің мақсатым, жалпыланған жылу теңдеуін, жылу полимдары арқылы табылған Лагерь полиномдарын қолдана отырып есепті шығару.

CONTENTS

I.INTRODUCTION

Problems for the heat equation in domains with moving boundaries including Stefan problem, are very important for many applications, such as melting and welding, freezing and cooling processes, displacement of oil by water and other phenomena.

The general method for the solution of these problems is based on the reduction of a boundary value problem to the integral equations. This method enables to prove theorems of existence and uniqueness of the solution, however it is not effective for the calculation of the real temperature fields. Therefore the elaboration of analytical and numerical methods of solution is very actual. This paper is an attempt to find the analytical solution of the Stefan problem using special functions, namely the integral error functions.

The integral error functions determined by recurrent formulas.

II. GENERALIZATION OF INTEGRAL ERROR FUNCTION

2.1 The Integral Error Functions

The integral error functions determined by recurrent formulas below

$$i^n erfc(x) = \int_x^\infty i^{n-1} erfc(\nu)d\nu, \quad n = 1, 2...$$

$$i^0 erfc(x) = erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-\nu^2)d\nu \qquad (2.1.1)$$

were introduced by Hartree in 1935.

We can obtain from (2.1.1)

$$i^n erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty (\nu - x)^n \exp(-\nu^2)d\nu \qquad (2.1.2)$$

These functions satisfy the differential equation

$$\frac{d^2}{dx^2}i^n erfc(x) + 2x\frac{d}{dx}i^n erfc(x) - 2ni^n erfc(x) = 0 \qquad (2.1.3)$$

and recurrent formulas

$$2ni^n erfc(x) = i^{n-2}erfc(x) - 2xi^{n-1}erfc(x) \qquad (2.1.4)$$

Integral error functions (some times they are called also Hartree functions) are very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation given below

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \qquad (2.1.5)$$

in a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, however functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc\frac{\pm x}{2a\sqrt{t}}$$

satisfy the equation (2.1.5) as well as their linear combination or even series

$$u(x,t)=\sum_{n=0}^{\infty}\left[A_n u_n(x,t)+B_n u_n(-x,t)\right]$$

for any constants A_n, B_n. We can choose to satisfy the boundary conditions at $x=0$ and $x=a(t)$, if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$. [2]

2.2 Properties of Integral Error Functions

Let us derive new properties of IEF, which are not considered in above-mentioned papers.

1) If is an integer, then

$$i^n erfc(-x)+(-1)^n i^n erfc(x)=\frac{1}{2^{n-1}n!i^n}H_n(ix)=\frac{1}{2^{n-1}n!}e^{-x^2}\frac{d^n}{dx^n}e^{x^2}$$

with $i=\sqrt{-1}$ and Hermite polynomials $H_n(x)$ in the right side . Indeed , using formula (2.1.2) we can write

$$i^n erfc(-x)+(-1)^n i^n erfc(x)=\frac{2}{\sqrt{(\pi)}}\frac{1}{n!}\int_x^{\infty}(v+x)^n exp(-v^2)dv+$$

$$+\frac{(-1)^n 2}{n!\sqrt{\pi}}\int_x^{\infty}(v-x)^n exp(-v^2)dv=\frac{2}{n!\sqrt{\pi}}\int_{-\infty}^{\infty}(v+x)^n exp(-v^2)dv=\frac{1}{2^{n-1}n!i^n}H_n(ix)$$

Using formula for Hermite polynomials we can derive

$$i^n erfc(-x)+(-1)^n i^n erfc(x)=\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2m}}{2^{2m-1}m!(n-2m)!} \qquad (2.2.1)$$

If n is even, $n=2k$, then

$$i^{2k} erfc(x)+i^{2k} erfc(-x)=\sum_{m=0}^{k}\frac{x^{2(k-m)}}{2^{2m-1}m!(2k-2m)!}$$

In particular

$erfc(x)+erfc(-x)=2$

$i^2 erfc(x)+i^2 erfc(-x)=1/2+x^2$

$i^4 erfc(x) i^4 erfc(-x)=1/8+1/4x^2+1/12x^4$

If $n = 2k + 1$, then

$$i^{2k+1} erfc(x) + i^{2k-1} erfc(-x) = \sum_{m=0}^{k} \frac{x^{2(k-m)+1}}{2^{2m-1} m!(2k-2m+1)!}$$

In particular

$ierfc(-x) - ierfc(x) = 2x$

$i^3 erfc(-x) - i^3 erfc(x) = 1/2x + 1/3x^3$

$i^5 erfc(-x) - i^5 erfc(x) = 1/16x + 1/24x^3 + 1/60x^5$

2) The proof of the formula

$$i^n erfc(-x) - i^n erfc(x) = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n}(e^{x^2} erfc(x)) \qquad (2.2.2)$$

where

$$erf(x) = 1 - erfc(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\nu^2) d\nu$$

can be obtained by mathematical induction method using recurrent formula (2.1.4).

3) Differentiating the right side of formula (2.2.2), we obtain

$$i^n erfc(-x) - (-1)^n i^n erfc(x) = P_n(x) erf(x) - Q_n(x) \frac{2}{\sqrt{\pi}} exp(-x^2). \qquad (2.2.3)$$

where polynomials $P_n(x)$ and $Q_n(x)$ are defined by formulas

$$P_n(x) = \sum_{m=0}^{\frac{n}{2}} \frac{x^{n-2m}}{2^{2m-1} m!(n-2m)!}, \quad Q_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k} H_{n-k-1}(x)}{2^{n-k}(n-k)!} P_k(x)$$

4) From (2.2.3), (2.2.4) we can obtain the explicit expressions for Hartree functions of an integer index

$$i^n erfc x = \frac{(-1)^n}{2} \left[P_n(x) erfc x + Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)\right] \qquad (2.2.4)$$

$$i^n erfc(-x) = \frac{1}{2} \left[P_n(x) erfc(-x) - Q_n(x) \frac{2}{\sqrt{\pi}} \exp(-x^2)\right] \qquad (2.2.5)$$

5) Using L'Hospital rule and representation (2.1.1), it is not difficult to show that

$$\lim_{x \to \infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!} \qquad (2.2.6)$$

6) Using property 2 one can derive following formula

$$u(x,t) = \sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^{m}\beta_{2n,m} + A_{2n-1}\sum_{m=0}^{n} x^{2n-2m-1}t^{m}\beta_{2n-1,m}\right] \quad (2.2.7)$$

Where $u(x,t)$ is Heat polynomial which exactly satisfy Heat Equation and

$$\beta_{n,m} = \frac{1}{2^{n-m-1}m!(n-2m)!} \text{ [3]}$$

2.3 Integral Error Function

$i^n erfc(x) = \int_x^{\infty} i^{n-1} erfc(v)dv,\ n = 1,2,\ldots$

$i^0 erfc(x) \equiv erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} exp(-v^2)dv.$ (1.1)

were introduced by Hartree [1] in 1935.

One can obtain from (1.1)

$i^n erfc(x) = \frac{2}{\sqrt{\pi}}\frac{1}{n!}\int_x^{\infty}(v-x)^n exp(-v^2)dv,$ (1.2)

They satisfy the differential equation

$\frac{\partial^2}{\partial x^2} i^n erfc(x) + 2x\frac{\partial}{\partial x} i^n erfc(x) - 2ni^n erfcx = 0,$ (2.3.1)

and recurrent formulas

$2ni^n erfc(x) = i^{n-2}erfc(x) - 2xi^{n-1}erfc(x).$ (2.3.2)

Integral error functions (sometimes they are called also Hartree functions) are very useful for investigation of heat transfer, diffusion and other phenomena, which can be described by the equation

$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ (2.3.3) in a region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$. since the functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc(\frac{\pm x}{2a\sqrt{t}})$$

satisfy the equation (2.3.3) as well as their linear combination or even series

$$u(x,t) = \sum_{n=0}^{\infty}(A_n u_n(x,t) + B_n u_n(-x,t))$$

for any constants A_n , B_n. We can choose these constants to satisfy the boundary conditions at x = 0 and $x = \alpha(t)$, if given boundary functions can be expanded into Taylor series with powers t or $\sqrt{t}$. [6]

Let us derive new properties of Hartree functions, which are not considered in above-mentioned papers.

1. If n is an integer, then

$$i^n erfc(-x) + (-1)^n i^n erfc(x) = \frac{1}{2^{n-1} n! i^n} H_n(ix) = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{\partial^n}{\partial x^n}(e^{x^2})$$

with $i = \sqrt{1}$ and Hermite polynomials $H_n(x)$ in the right hand side. Indeed, using formula (2.3.1) one can write

$$i^n erfc(-x) + (-1)^n i^n erfc(x) = \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_{-x}^{\infty} (v + x)^n exp(-v^2) dv +$$

$$+ \frac{2}{\sqrt{\pi}} \frac{(-1)^n}{n!} \int_{x}^{\infty} (v - x)^n exp(-v^2) dv = \frac{2}{\sqrt{\pi} n!} \int_{-\infty}^{\infty} (v + x)^n exp(-v^2) dv =$$

$$= \frac{1}{2^{n-1} n! i^n} H_n(ix)$$

Using formula for Hermite polynomials one can derive

$$i^n erfc(-x) + (-1)^n i^n erfc(x) = \sum_{m=0}^{\frac{n}{2}} \frac{x^{n-2m}}{2^{2m-1} m! (n-2m)!} \quad (2.3.4)$$

If n = 2k, then

$i^{2k} erfc(x) + i^2 kerfc(-x) = \sum_{m=0}^{k} = \frac{x^{2(k-m)}}{2^{2m-1} m!(2k-2m)!}$

i.c.

$erfc(x) + erfc(-x) = 2$

$i^2 erfc(x) + i^2 erfc(-x) = \frac{1}{2} + x^2$

$i^4 erfc(x) + i^4 erfc(-x) = \frac{1}{16} + \frac{x^2}{4} + \frac{x^4}{12}$

$i^2 kerfc(x) + i^2 kerfc(-x) = \frac{1}{2^{2k-1} k!} + \frac{x^2}{2^{2k-3}(k-1)!2!} + \frac{x^4}{22k-5(k-2)!4!} + \ldots + \frac{x^{2k}}{2^{-1}(2k!)}$

If n = 2k + 1, then

$i^{2k-1} erfc(-x) - i^{2k+1} erfc(x) = \sum_{m=0}^{k} = \frac{x^{2(k-m+1)}}{2^{2m-1} m!(2k-2m+1)!}$

i.e.

$ierfc(-x) - ierfc(x) = 2x$

$i^3 erfc(-x) - i^3 erfc(x) = \frac{x}{2} + \frac{x^3}{3}$

$i^5 erfc(-x) - i^5 erfc(x) = \frac{x}{2^3 \cdot 2!} + \frac{x^3}{2 \cdot 3!} + \frac{2x^5}{5!}$

$i^{2k-1} erfc(-x) - i^{2k+1} erfc(x) = \frac{x}{2^{2k-1} . k!} + \frac{x^3}{2^{2k-3} . (k-1)! \cdot 3!} + \frac{x^5}{2^{2k-3} . (k-2)! \cdot 5!} +$

$\ldots + \frac{x^{2k-1}}{2^{-1} . (2k+1)!}$

2. The proof of the formula

$i^n erfc(-x) + (-1)^n i^n erfc(x) = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{\partial^n}{\partial x^n} (e^{x^2} erfx)$ (2.3.5)

where

$erf(x) = 1 - erfc(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp(-v^2) dv,$

can be obtained by mathematical induction method with the help of the recurrent formula (2.3.3).

3. Differentiating the right hand side of formula (2.3.6) we obtain

$$i^n erfc(-x) + (-1)^n i^n erfc(x) = P_n(x) erf(x) - Q_n \frac{2}{\sqrt{\pi}} exp(-x^2). \quad (2.3.7)$$

where polynomials $P_n(x)$ and $Q_n(x)$ are defined by formulas

$$P_n(x) = \sum_{m=0}^{\frac{n}{2}} \frac{x^{n-2m}}{2^{2m-1} \cdot m! \cdot (n-2m)!}. \qquad Q_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k} H_{n-k-1}(x)}{2^{n-k} \cdot (n-k)!} \ [5]$$

Heat polynomials can be shown in terms of the integral error functions using the expressions

$$P_n(x,t) = (2a\sqrt{t})^n [i^n erfc\frac{(}{-x}2a\sqrt{t} + (-1)^n i^n erfc\frac{x}{2a\sqrt{t}}] = \sum_{m=0}^{[\frac{n}{2}]} C_{n,m} x^{n-2m} \cdot t^m \quad (2.3.8)$$

where $C_{n,m} = \frac{2a^{2m}}{m!(n-2m)!}$

If we replace the time variable t by the new variable $a^2 t$,then we can put in above expressions a=1. If we multiply all heat polynomials by the factor $\frac{n!}{2}$, then the coefficient at x^n becomes equal to 1. It is more convenient at calculations. Thus we can define the heat polynomials by the expression (1) where

$$C_{n,m} = \frac{n!}{m!(n-2m)!} \quad (2.3.9)$$

The first heat polynomials are

$P_0(x,t) = 1 \qquad P_1(x,t) = x$

$P_2(x,t) = x^2 + 2t \qquad P_3(x,t) = x^3 + 6xt$

$P_4(x,t) = x^4 + 12x^2t + 12t^2 \qquad P_5(x,t) = x^5 + 20x^3t + 60xt^2$

2.4 Generating Functions

The heat polynomials can be introduced also using the generating function

$$g(x,t,z) = e^{xz+tz^2}$$

This function satisfies the heat equation $g_t = g_{xx}$ for any z. The coefficients $v_n(x,t)$ in the Taylor series

$$g(x,t,z) = \sum_{n=0}^{\infty} v_n(x,t) z^n \quad (2.4.1)$$

are also solutions of the heat equation

$$\frac{\partial v_n}{\partial t} = \frac{\partial^2 v_n}{\partial x^2}$$

It can be checked directly by differentiating of the equation (2.4.1) with respect to x and t.

$$\frac{\partial g}{\partial x} = e^{xz-tz^2} z = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} v_n(x,t) \frac{z^n}{n!} = \sum_{n=0}^{\infty} v_n(x,t) \frac{z^{n+1}}{n!}$$

$$\frac{\partial g}{\partial t} = e^{xz+tz^2} z = \sum_{n=0}^{\infty} \frac{\partial}{\partial t} v_n(x,t) \frac{z^n}{n!} = \sum_{n=0}^{\infty} v_n(x,t) \frac{z^{n-2}}{n!}$$

By comparing coefficients we obtain

$$\frac{\partial}{\partial x} v_n(x,t) = n v_{n-1}(x,t), \qquad n = 1, 2, \ldots,$$

$$\frac{\partial}{\partial t} v_n(x,t) = n(n-1) v_{n-2}(x,t) = \frac{\partial^2}{\partial x^2} v_n(x,t), \qquad n = 2, 3, \ldots,$$

The explicit expressions for the function $v_n(x,t)$ can be obtained by the product of two power series

$e^{xz} = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{x^n}{n!}$

and

$e^{tz^2} = \sum_{n=0}^{\infty} b_n z^n, \qquad b_n = \frac{t^n}{n!}$

Using the Cauchy's rule this product can be written in the form

$e^{xz-tz^2} = \sum_{n=0}^{\infty} c_n z^n; \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Thus

$v_n(x,t) = n! c_n = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{t^k}{k!} \frac{x^{n-2k}}{(n-2k)!}$

which coincides with (1),(2).

It is very important for the applications that

$v_n(x,0) = x^n, \qquad v_{2n}(0,t) = \frac{(2n)!t^n}{n!}, \qquad v_{3n-1}(0,0) = 0$

It should be noted that sometimes the application of integral error functions has an advantage in comparison with the heat polynomials, in particular for satisfying a conditions at infinity, when the formula

$\lim_{x\to\infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!}$

plays very important role.

2.5 Analytical solution of One Phase Stefan Problem

The first analytical solution of one phase Stefan problem, which describes the dynamics of soil freezing has been published by Lame and Clayperon. Solution of two phase Stefan problem was represented by Stefan. Solutions of these problems were obtained for $\alpha(t) = \alpha\sqrt{(t)}$ case and some auotomodel cases.

Despite the quite extensive list of problems in literature which lead to the necessity to solve Stefan type problems see: e.g., and a long bibliography on methods for solving these problems lead to additional difficulties which occur due to the degeneracy of domains. In some specific cases particularly for free moving boundaries it is possible to construct Heat potentials and a problem can be reduced to the system of integral equations [6], however in the case of degeneracy, singularity in integral equations occur, and method of successive approximations is inapplicable in general. Moreover, the use of numerical methods is problematic when the number of parameters is great. Therefore, development of new analytical methods is very important especially for various applications because it enables one to analyze an interrelationship of different input parameters and their influence on the dynamics of investigating phenomena.

As for applications: a wide range of electric contact phenomena, in particular, the phenomena occurring at the interaction of electrical arc with electrode can be described in dynamic use of the presented method see e.g., for very short arc duration (nanosecond diapason), when experimental investigation is very difficult. In this study we will try to find solution of one Phase Stefan problem for degenerate domain with $\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^{\frac{n}{2}}$ moving boundary.

Tracking answers of these questions will be organized as following. In the continuation of this section Integral Error Functions and its properties necessary for elaboration of new methods are presented. In subsection 1.4 a test problem is solved by proposed method. In sections two and three one phase Stefan problem, its analytical solution and convergence of series represented. For finding analytical solution we mainly follow the method proposed by S.N. Kharin in applying Faa Di Brunos formula for Integral Error Functions. For Heat polynomials we utilize Newtons polynomial (generalization of Newtons binomial) and its multinomial coefficients.

Heat transfer between two bars that have ideal contact is described by the equations:

$$\frac{\partial U}{\partial t} = a^2\frac{\partial^2 U}{\partial t^2}.\ 0 < x < \infty.\ (2.5.1)$$

with the initial conditions:

$$U(0.0) = 0 \quad (3.2.2) \quad U(x.0) = f(x) \quad (2.5.2)$$

conditions of conjugations of temperature and heat flux

$$-\lambda U(0.t) = P(t) \qquad (2.5.3)$$

$$U(\alpha\sqrt{t}.t) = U_m \qquad (2.5.4)$$

$$-\lambda\frac{\partial U}{\partial x}|_{x=\alpha\sqrt{t}} = L\gamma\frac{d\alpha}{dt} \qquad (2.5.5)$$

Suggesting that initial function and function of heat flux can be expanded in Maclaurin series

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.\quad P(t) = \sum_{n=0}^{\infty}\frac{p^{(n)}(0)}{n!}(\sqrt{t})^n \qquad (2.5.6)$$

we represent the solution in the form

$$U(x.t) = \sum_{n=0}^{\infty}(2a\sqrt{t})^n\left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right] \qquad (2.5.7)$$

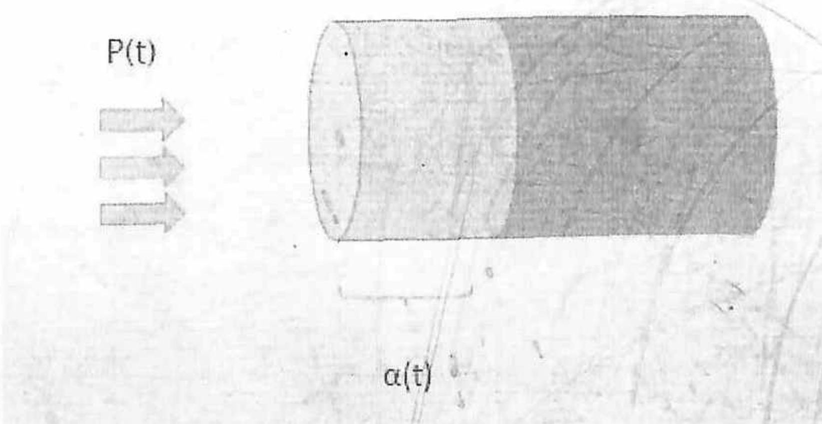

Figure (2.5.8) Boundary between two (liquid and solid) phases

$$\frac{\partial U}{\partial t} = a^2\frac{\partial U^2}{\partial x^2}, \qquad 0 < x < \alpha(t)$$

$U(0,0) = 0$ (2.5.9)

$U(0,t) = P(t)$ (2.5.10)

$U(\alpha(t),t) = U_m$ (2.5.11)

$-\lambda\frac{\partial U}{\partial x}|_{x=\alpha(t)} = L_2\frac{d\alpha}{dt}$ (2.5.12)

Solution:

$$U(x,t) = \sum_{n=0}^{\infty}(2a\sqrt{t})^n\left[A_n i^n erfc\frac{x}{2a\sqrt{t}} + B_n i^n erfc\frac{-x}{2a\sqrt{t}}\right] \qquad (2.5.13)$$

Lemma:

1) $\lim_{x\to\infty}\frac{i^n erfc(-x)}{x^n} = \frac{2}{n!}$

2) $\lim_{t\to 0}(2a_2\sqrt{t})^n i^n erfc\frac{x}{2a_2\sqrt{t}} = 0$

Using (2.5.13), from condition (2.5.10), we get

$$-\lambda U(0,t) = -\lambda\sum_{n=0}^{\infty}(2a\sqrt{t})^n[A_n i^n erfc0 + B_n i^n erfc0] = P(t)$$

$$-\lambda U(0,t) = -\lambda\sum_{n=0}^{\infty}(2a\sqrt{t})^n[A_n + B_n]erfc0 = P(t) = \sum_{n=0}^{\infty}\frac{P^{(n)}(0)}{n!}(\sqrt{t})^n$$

By denoting $P_n = \sum_{n=0}^{\infty}\frac{P^{(n)}(0)}{n!}(\sqrt{t})^n$, we have left

$A_n = -\frac{P_n}{\lambda(2a)^n i^n erfc0} - B_n$, by substituting $\frac{P_n}{\lambda(2a)^n i^n erfc0}$ to the β_n, we get

$$A_n = -\beta_n - B_n \qquad (2.5.14)$$

At the condition n=0:

$$A_0 = -\beta_0 - B_0 \qquad (2.5.15)$$

From condition (2.5.11), using $\alpha(t) = \alpha\sqrt{t}$, we have

$$U(\alpha\sqrt{t}, t) = \sum_{n=0}^{\infty}(2a\sqrt{t})^n[A_n i^n erfc\frac{\alpha}{2a} + B_n i^n erfc\frac{-\alpha}{2a}] = U_m$$

n=0:

$$[A_0 erfc\frac{\alpha}{2a} + B_0 i^0 erfc\frac{-\alpha}{2a}] = U_m$$

By using (3.3.7) condition, we get

$$(\beta_0 - B_0)erfc\frac{\alpha}{2a} + B_0 i^0 erfc\frac{-\alpha}{2a}] = U_m$$

$$B_0(erfc\frac{-\alpha}{2a} - erfc\frac{\alpha}{2a}) = U_m + \beta_0 erfc\frac{\alpha}{2a}$$

$$B_0 = \frac{\beta_0 erfc\frac{\alpha}{2a}}{(erfc\frac{-\alpha}{2a} - erfc\frac{\alpha}{2a})} \tag{2.5.16}$$

From condition (2.5.12), by taking the derivative of (2.5.13), we get

$$-\lambda\sum_{n=0}^{\infty}(2a\sqrt{t})^{n-1}\left[-A_n i^{n-1}erfc\frac{\alpha}{2a} + B_n i^{n-1}erfc\frac{-\alpha}{2a}\right] = L\gamma\frac{\alpha}{2\sqrt{t}}$$

For the condition n=0:

$$-\lambda\frac{1}{2a}[-A_0 + B_0]\frac{2}{\sqrt{\pi}}e^{\frac{-\alpha^2}{4a^2}} = L\gamma\frac{\alpha}{2}$$

By using conditions (2.5.15), we get

$$-\frac{\lambda}{2}[2B_0 + \beta_0]\frac{2}{\sqrt{\pi}} = L\gamma\alpha e^{\frac{\alpha^2}{4a^2}}$$

And using condition (2.5.16), we finally get our equation, which consists only from α and coefficients

$$-\frac{\lambda}{2}\beta_0[\frac{2erfc\frac{\alpha}{2a}}{(erfc\frac{-\alpha}{2a} - erfc\frac{\alpha}{2a})}] = L\gamma\alpha e^{\frac{\alpha^2}{4a^2}} \tag{2.5.17}$$

$a = 1$

$\lambda = 1$

$\beta_0 = 107$

$U_m = 100$

$L = 50$

$\gamma = 2$

Firstly, we find β_0, however it depends only from coefficients as P_0, λ, and a to which we have given numbers. After finding β_0, we come to the last equation (2.5.17) that we obtained. There are two equation which are equal to each other and depends only from α that we should find. We denote left side of equation as $f(\alpha)$, and right side as $g(\alpha)$.By the way we use approximation method by Matlab which code shown in Figure(2.5.9).

```
a=1;
lambda=1;
p0=107;
Um=100;
l=80;
gamma=2;


beta=p0/(lambda);


for i=1:100
    x=i/100;
    y1(i)=-(lambda^2)/(a*sqrt(pi))*((1/(1-erfc(x/2)))*(Um-beta*erfc(x/2))-beta)
    y2(i)=l*gamma*x*exp(x.^2/4);

end

plot(1/100*(1:100),y1,'[illegible]');
hold on
plot(1/100*(1:100),y2,'[illegible]');
grid on
```

Figure (2.5.18) Code of equation (2.5.17)

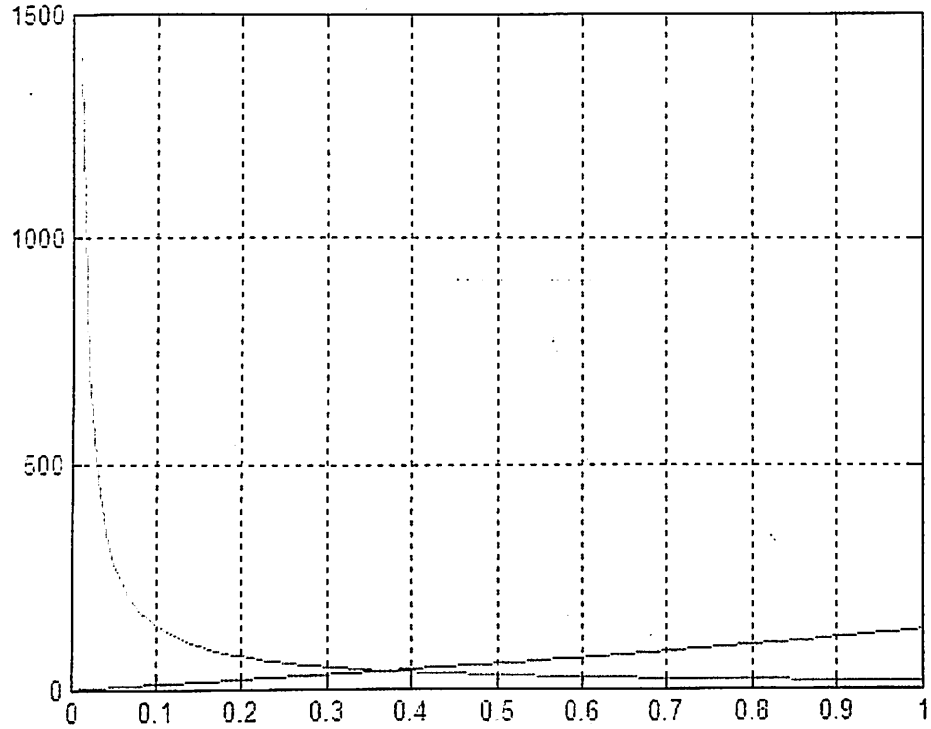

Figure (2.6.19) Graph view of solution of the equation(2.5.17)

[2]

III.LAGUERRE POLYNOMIALS

3.1 Some identities for the generalized Laguerre polynomials

The summed up Laguerre polynomials $L_n^{(\alpha)}(x)$ related with non-negative integer n and genuine number $\alpha > -1$ are generally utilized in numerous issues of mathematical material science and quantum mechanics, for instance, in the incorporation of Helmholtzs condition in paraboloidal directions, in the hypothesis of the spread of electro-attractive motions along long queues, and so on., just as in material science in association with the arrangement of the second-request straight differential condition:

$x y'' + (\alpha + 1 - x)y' + ny = 0$ (3.1.1)

These polynomials fulfill some recurrence relations. One extremely valuable, when extricating properties of the wave elements of the hydrogen molecule, is the accompanying three-term recurrence relation

$L_{n+1}^{(\alpha)}(x) = \frac{2n+1-\alpha-x}{n+1} L_n^{(\alpha)}(x) - \frac{n-\alpha}{n+1} L_{n-1}^{(\alpha)}(x), (n \geq 1)$ (3.1.2)

with the initial conditions $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = 1 + \alpha - x$. Specifically, the case $\alpha = 1$ in (3.1.2)ives the old style Laguerre polynomials $L_n(x)$ fulfilling

$L_{n+1}^{(\alpha)}(x) = \frac{2n+2-x}{n+1} L_n^{(\alpha)}(x) - L_{n-1}(x), (n \geq 1)$ (3.1.3)

with the initial conditions $L_0(x) = 1$ and $L_1(x) = 2 - x$.

TThis group of generalized Laguerre polynomials structure a total symmetrical framework in the weighted Sobolev space $L^2_{\Omega_\alpha}(R^+)$ with the weighted function $\Omega_\alpha(x) = x^\alpha e^{-x}$, as follows

$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{\Gamma(n-\alpha+1)}{n!} \delta_{mn}, (m, n \geq 0)$ (3.1.4)

where Γ is the Gamma function and δ_{ij} is the Kronecker delta symbol given by $\delta_{ij} = 1$ or 0 according to i = j or i $\neq$ j . In fact, the generalized Laguerre polynomials are eigenfunctions of the Sturm-Liouville problem:

$$\chi^{-\alpha}e^{\chi}(\chi^{\alpha-1}e^{-\chi}(L^{(\alpha)}_{n}(\chi))')' + \mu_n L_n^{(\alpha)}(\chi) = 0, (n \geq 0). (3.1.5)$$

with the eigenvalues $\mu_n = n$. Besides, we really have the accompanying closed formula for the generalized Laguerre polynomials:

$$L_n^{(\alpha)}(\chi) = \sum_{k=0}^{n}(-1)^k \binom{n+\alpha}{n-k} \frac{\chi^k}{k!}, (n \geq 0). (3.1.6)$$

where $\binom{\gamma}{k}$ is the binomial coefficients given by

$$\binom{\gamma}{0} = 1 \text{ and } \binom{\gamma}{k} = \frac{\gamma(\gamma-1)(\gamma-2)\ldots(\gamma-k+1)}{k(k-1)(k-2)\ldots1}. (3.1.7)$$

for positive integer k and complex number γ. It is notable that the recipe (3.1.6) stems from Rodrigues formula for the generalized Laguerre polynomials:

$$L_n^{\alpha}(\chi) = \frac{\chi^{-\alpha}e^{\chi}}{n!} \cdot \frac{\partial^n}{\partial\chi^n}(e^{-\chi}\chi^{n-\alpha}) = \chi^{-\alpha}\frac{(\frac{\partial}{\partial\chi}-1)^n}{n!}\chi^{n-\alpha}, (n \geq 0). (3.1.8)$$

However, for the closed equation (3.1.6), it appears that none has considered it yet, in any event we have not seen any related outcomes previously. The equation (3.1.6) is exceptionally intriguing, in light of the fact that it uncovers great worth conveyances of the generalized Laguerre polynomials. [4]

Theorem 3.1.

Let l, m, n be non-negative integers with $0 \leq l \leq m+n$. Then

$$\sum_{k=0}^{l}\binom{m}{k}(m+n-k)!\chi^k L_{(l-k)}^{(\alpha+k)}(\chi) = \sum_{k=0}^{l}\binom{m}{k}(m+n-k)!(-\chi)^k\binom{l+\alpha}{l-k},$$

(3.1.9)

It pursues that we demonstrate some extraordinary instances of Theorem 3.1. It is effectively observed that positive whole numbers n, k with $k > n$. In this manner, by taking l = m + n in Theorem 3.1, we have

$$\sum_{k=0}^{m}\binom{m}{k}(m+n-k)!\chi^k L_{(m+n-k)}^{(\alpha+k)}(\chi)$$

$$= \sum_{k=0}^{m} \binom{m}{k} (m+n-k)!(-x)^k \binom{m+n+\alpha}{m+n-k}, (m,n \geq 0).(3.1.10)$$

The case n = 0 in (3.1.10) gives

$$\sum_{k=0}^{m} \binom{m}{k} (m-k)! x^k L_{m-k}^{(\alpha+k)}(x) = m! \binom{m+\alpha}{m}, (m \geq 0). (3.1.11)$$

If we take m = 0 in (3.1.10), we get

$$n! L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(-x)^k \binom{n+\alpha}{n-k}, (n \geq 0). (3.1.12)$$

which together with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for non-negative integer n, k with $k \leq n$.
It is intriguing to call attention to that (3.1.11) is extremely simple to the backwards recipe of the generalized Laguerre polynomials, to be specific

$$x^m = m! \cdot \sum_{k=0}^{m} \binom{m+\alpha}{m-k} (-1)^k L_k^{(\alpha)}(x), (m \geq 0), (3.1.13)$$

3.2 The heat polynomials for the Generalized Heat Equation

The heat polynomials are very useful for the solution of value-boundary problems for the heat equation. They can be derived by different means[6].

The heat polynomials are introduced from the solution of the heat equation in terms of the integral error functions using the expressions

$$P_n(x,t) = (2a\sqrt{t})^n [i^n erfc \tfrac{-x}{2a\sqrt{t}} + (-1)^n i^n erfc \tfrac{x}{2a\sqrt{t}}] = \sum_{m=0}^{[\frac{n}{2}]} C_{n,m} x^{n-2m} t^m \quad (3.2.1)$$

where $\quad C_{n,m} = \frac{2a^{2m}}{m!(n-2m)!}$

If we replace the time variable by the new variable ,then we can put in above expressions . If we multiply all heat polynomials by the factor , then the coefficient at becomes equal to 1. It is more convenient at calculations. Thus w can define the heat polynomials by the expression (2.1.1) where

$$C_{n,m} = \frac{n!}{m!(n-2m)!} \quad (3.2.2)$$

The first heat polynomials are

$P_0(x,t) = 1, \quad P_1(x,t) = x, \quad P_2(x,t) = x^2 + 2t, \quad P_3(x,t) = x^3 + 6xt$

$P_4(x,t) = x^4 + 12x^2t + 12t^2, \quad P_5(x,t) = x^5 + 20x^3t + 60xt^2$

There exists the relationship between integral error functions and heat potentials. If a function $\varphi(t)$ is an analytical function then integrating by parts the potential of the double layer we get

$$\int_0^t \frac{x e^{-\frac{x^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi}(t-\tau)^3}\varphi(\tau)d\tau = -\int_0^t \varphi(\tau)d(erfc\frac{x}{2a\sqrt{t-\tau}}) = \varphi(0)erfc\frac{x}{2a\sqrt{t}}+$$

$$+\int_0^t \varphi'(\tau)erfc\frac{x}{2a\sqrt{t-\tau}}d\tau = \varphi(0)erfc\frac{x}{2a\sqrt{t}} - \int_0^t \varphi'(\tau)d[4(t-\tau)i^2erfc\frac{x}{2a\sqrt{t-\tau}}] =$$

$$= \varphi(0)erfc\frac{x}{2a\sqrt{t}} + \varphi'(0)4ti^2erfc\frac{x}{2a\sqrt{t}} + \int_0^t \varphi''(\tau)4(t-\tau)i^2erfc\frac{x}{2a\sqrt{t-\tau}}d\tau = \quad (3.2.3)$$

$$= \ldots = \sum_{n=0}^{\infty}\varphi^{(n)}(0)\frac{t^n i^{2n}erfc\frac{x}{2a\sqrt{t}}}{n!i^{2n}erfc0} = \sum_{n=0}^{\infty}\varphi^{(n)}(0)(4t)^n i^{2n}erfc\frac{x}{2a\sqrt{t}}$$

Similarly the integral responsible for the initial condition

$$I = \int_0^{\infty}[e^{-\frac{(x-\xi)^2}{4a^2t}} \pm e^{-\frac{(x+\xi)^2}{4a^2t}}]\frac{f(\xi)}{2a\sqrt{\pi t}}d\xi \quad (3.2.4)$$

for the analytical function $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ can be written in the form

$$I = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left[\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2a\sqrt{t}}}^{\infty} e^{-z^2}(x+2az\sqrt{t})^n dz \pm \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-z^2}(-x+2az\sqrt{t})^n dz\right]$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(2a\sqrt{t})^n\left[\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2a\sqrt{t}}}^{\infty} e^{-z^2}(\frac{x}{2a\sqrt{t}}+z)^n dz \pm \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-z^2}(-\frac{x}{2a\sqrt{t}}+z)^n dz\right] =$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(2a\sqrt{t})^n\left[i^n erfc\frac{-x}{2a\sqrt{t}} \pm i^n erfc\frac{x}{2a\sqrt{t}}\right] \quad (3.2.5)$$

The heat polynomials can be introduced also using the generating function

$$g(x,t,z) = e^{xz - tz^2}$$

This function satisfies the heat equation $g_t = g_{xx}$ for any z. The coefficients $v_n(x,t)$ in the Taylor series

$$g(x,t,z) = \sum_{n=0}^{\infty} v_n(x,t) z^n \qquad (3.2.6)$$

are also solutions of the heat equation

$$\frac{\partial v_n}{\partial t} = \frac{\partial^2 v_n}{\partial x^2}$$

It can be checked directly by differentiating of the equation (3.2.6) with respect to x and t.

$$\frac{\partial g}{\partial x} = e^{xz-tz^2} z = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} v_n(x,t) \frac{z^n}{n!} = \sum_{n=0}^{\infty} v_n(x,t) \frac{z^{n+1}}{n!}$$

$$\frac{\partial g}{\partial t} = e^{xz-tz^2} z = \sum_{n=0}^{\infty} \frac{\partial}{\partial t} v_n(x,t) \frac{z^n}{n!} = \sum_{n=0}^{\infty} v_n(x,t) \frac{z^{n+2}}{n!}$$

By comparing coefficients we obtain

$$\frac{\partial g}{\partial x} v_n(x,t) = n v_{n-1}(x,t) \qquad n = 1, 2, \ldots$$

$$\frac{\partial g}{\partial x} v_n(x,t) = n(n-1) v_{n-2}(x,t) = \frac{\partial^2}{\partial x^2} v_n(x,t) \qquad n = 2, 3, \ldots$$

The explicit expressions for the functions $v_n(x,t)$ can be obtained by the product of two power series

$$e^{xz} = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{x^n}{n!}$$

$$e^{tz^2} = \sum_{n=0}^{\infty} b_n z^n, \qquad b_n = \frac{t^n}{n!}$$

Using the Cauchys rule this product can be written in the form

$$e^{xz-tz^2} = \sum_{n=0}^{\infty} c_n z^n, \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Thus

$$v_n(x,t) = n! c_n = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{t_k}{k!} \frac{x^{n-2k}}{(n-2k)!} \qquad (3.2.7)$$

which coincides with (3.2.1), (3.2.2).

It is very important for applications that

$$v_n(x,0) = x^n \quad v_{2n}(0,t) = \frac{(2n)! t^n}{n!} \quad v_{2n+1}(0,t) = 0 \quad (3.2.8)$$

It should be noted that sometimes the application of integral error functions has an advantage in comparison with the heat polynomials, in particular for satisfying a conditions at infinity, when the formula

$$\lim_{x\to\infty} \frac{i^n erfc(x)}{x^n} = \frac{2}{n!} \quad (3.2.9)$$

plays very important role.

Example.

Let us consider the problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \alpha t, \quad t > 0 \quad (3.2.10)$$

with the boundary conditions

$$u(0,t) = f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \quad (3.2.11)$$

$$u(\alpha t, t) = g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n \quad (3.2.12)$$

$$u(\alpha t, t) = g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n$$

The solution of this problem can be found in the form

$$u(x,t) = \sum_{n=0}^{\infty} A_n v_{2n}(x,t) + \sum_{n=0}^{\infty} B_n v_{2n+1}(x,t) \quad (3.2.13)$$

Satisfying the boundary condition (3.2.11) and using (3.2.9) we obtain

$$\sum_{n=0}^{\infty} A_n \frac{(2n)!}{n!} t^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n.$$

$$A_n = \frac{f^{(n)}(0)}{(2n!)} \quad (3.2.14)$$

(14) From the second boundary condition (3.2.11) we get

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} A_n v_{2n}(\alpha t, t) + \sum_{n=0}^{\infty} B_n v_{2n-1}(\alpha t, t) \quad (3.2.15)$$

From the expression (2.1.7) we obtain

$$v_{2n}(\alpha t, t) = (2n)! \sum_{k=0}^{\infty} A_k \frac{\alpha^{2n-2k}}{k!(2n-2k)!} t^{2n-k} =$$

$$= (2n)! \sum_{m=0}^{\infty} A_{n-m} \frac{\alpha^{2m}}{k!(2m)!} t^{n-m} \quad (3.2.16)$$

$$v_{2n+1}(\alpha t, t) = (2n)! \sum_{k=0}^{\infty} B_k \frac{\alpha^{2n+1-2k}}{k!(2n+1-2k)!} t^{2n-k} =$$

$$= (2n)! \sum_{m=0}^{\infty} B_{n-m} \frac{\alpha^{2m+1}}{k!(2m+1)!} t^{n+m+1} \quad (3.2.17)$$

Substituting these expressions into (3.2.15) and compare coefficient at t according to recurrent formulas we get the values B_n [6].

Heat polynomials for the generalized heat equation

Confluent hypergeometric function and corresponding special functions

Successful applications of the heat polynomials for the solution of the standard heat equation induce the question: is it possible to find and to use similar functions for solution of other equations? To answer this question let us consider the equation

$$x\frac{d^2\varphi}{dx^2} + (\frac{\nu+1}{2} - x)\frac{d\varphi}{dx} + \frac{\beta}{2}\varphi = 0, \qquad \nu = 0, \qquad -\infty < \beta < \infty$$

It is well known that this equation has two linearly independent solutions

$$\varphi_1(x) = \Phi(-\tfrac{\beta}{2}, \tfrac{\nu-1}{2}; x), \qquad \varphi_2(x) = x^{\frac{1-\nu}{2}}\Phi(-\tfrac{1-\beta-\nu}{2}, \tfrac{3-\nu}{2}; x) \quad (3.2.18)$$

where $\Phi(a; b; x)$ is the confluent (degenerate) hypergeometric function . Setting $T(z) = \varphi(x)$, where $x = -z^2$, one can find that $T(z)$ satisfies the equation

$$\frac{d^2T}{dz^2} + (\nu + 2z)\frac{dT}{dz} - 2\beta T(z) = 0$$

Using this equation one can check up that the function

$$\Theta(z, t) = (2a\sqrt{t})^{\beta}T(\tfrac{z}{2a\sqrt{t}})$$

satisfies the equation

$$\frac{\partial\theta}{\partial t} = a^2(\frac{\partial^2\theta}{\partial z^2} + \frac{\nu}{z}\frac{\partial\theta}{\partial z}) \quad (3.2.19)$$

Hence the functions satisfies the equation

$$\frac{\partial\theta}{\partial t} = a^2(\frac{\partial^2\theta}{\partial z^2} + \frac{\nu}{z}\frac{\partial\theta}{\partial z}). \qquad (3.2.19)$$

Hence the function

$$S^1_{\beta,\nu}(z, t) = (2a\sqrt{t})^{\beta}\Phi(-\frac{\beta}{2}, \frac{\nu+1}{2}; -\frac{z^2}{4a^2t}) \qquad (3.2.20)$$

$$S^2_{\beta,\nu}(z, t) = (2a\sqrt{t})^{\beta}(\frac{z^2}{4a^2t})\Phi(-\frac{1-\nu-\beta}{2}, \frac{3-\nu}{2}; -\frac{z^2}{4a^2t})$$

satisfy the equation (3.2.19). Their linear combination can be written using the degenerate hypergeometric function of the second kind $\Psi(a, b, x)$:

$$\Gamma_{\beta,\nu}(z, t) = \frac{\Gamma(\frac{1-\nu}{2})}{\Gamma\frac{1-\nu-\beta}{2}}S^1_{\beta,\nu}(z, t) + \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{-\beta}{2})}S^2_{\beta,\nu}(z, t) = (2a\sqrt{t})^{\beta}\Psi(-\frac{\beta}{2}, \frac{\nu+1}{2}; -\frac{z^2}{4a^2t})$$

It is convenient to use the function $S_{\beta,\nu}(z,t)$ for problems with bounded regions, while the function $V_{\beta,\nu}(z,t)$ are applicable to unbounded regions.
Using the integral representation for the degenerate hypergeometric function

$$\Phi(\frac{-\beta}{2},\mu;-z^2)=\frac{2\Gamma(\mu)}{\Gamma(\mu+\frac{\beta}{2})}exp(-z^2)z^{-\mu-1}\int_0^{\infty}exp(-x^2)x^{\mu-\beta}I_{\mu-1}(2zx)dx \quad (3.2.21)$$

it is possible to show that

$$\lim\frac{1}{z^{\beta}}\Phi(\frac{-\beta}{2},1;-z^2)=\frac{1}{\Gamma(1+\frac{\beta}{2})} \quad (3.2.22)$$

This formula is a generalizatiom of formula (3.2.9). Let us consider the special cases.

1^0. **Plane case** (v=0)

Using formula (3.2.21) and evaluating integral in the right side one can show that

$$\Phi(\frac{-\beta}{2},\frac{1}{2};-x^2)=\frac{\pi}{2}\frac{\Gamma(\beta+1)}{\Gamma(\frac{\beta+1}{2})}[i^{\beta}erfc(-x)+i^{\beta}erfcx] \quad (3.2.23)$$

$$\Phi(\frac{1-\beta}{2},\frac{3}{2};-x^2)=\frac{\sqrt{\pi}}{4}\frac{\Gamma(\beta+1)}{\Gamma(\frac{\beta}{2}+1)}\frac{1}{x}[i^{\beta}erfc(-x)-i^{\beta}erfcx] \quad (3.2.24)$$

i.e the hypergeometric function can be represented by the plane Hartree functions, in this case, but the functions [6]

$$S^1_{\beta,0}(z,t)=\frac{\sqrt{\pi}}{2}\frac{\Gamma(1+\beta)}{\Gamma(\frac{\beta+1}{2})}(2a\sqrt{t})^{\beta}[i^{\beta}erfc\frac{-z}{2a\sqrt{t}}+i^{\beta}erfc\frac{z}{2a\sqrt{t}}] \quad (3.2.25)$$

$$S^2_{\beta,0}(z,t)=\frac{\sqrt{\pi}}{4}\frac{\Gamma(1+\beta)}{\Gamma(\frac{\beta}{2}+1)}(2a\sqrt{t})^{\beta}[i^{\beta}erfc\frac{-z}{2a\sqrt{t}}-i^{\beta}erfc\frac{z}{2a\sqrt{t}}] \quad (3.2.26)$$

satisfy the ordinary heat equation.

Cylindrical case (v=1)

In this case both functions (3.2.20) coincide:

$$S^1_{\beta,0}(z,t)=S^2_{\beta,0}(z,t)=(2a\sqrt{t})^{\beta}\Phi(\frac{-\beta}{2},1;-\frac{z^2}{4a^2t})$$

Expanding the hypergeometric function in the series

$$\Phi(-\frac{\beta}{2}, 1; -x^2) = \Gamma(\frac{\beta}{2}+1)\Sigma\frac{x^{2k}}{(k!)\Gamma(\frac{\beta}{2}+1-k)},$$

we can conclude that if $\beta = 2n$ is an even integer , then this series transforms into Laggurc polynomials

$$\Phi(-n, 1; -x^2) = \frac{n!}{(2n)!}L_n(-x^2) \qquad (3.2.27)$$

where

$$L_n(z) = \frac{1}{n!}e^{z}\frac{d^n}{dz^n}(e^{-z}z^n)$$

are the Laguerre polynomials.

Thus for an even index partial solution of the cylindrical heat equation

$$S^k_{2n,1}(z,t) = \frac{n!}{(2n)!}(4a^2t)L_n(-\frac{z^2}{4a^2t}) \qquad (3.2.28)$$

Spherical case (ν=2)

This the functions (*) are a generalization of the integral error functions of Hartee for any $r > 0$.

The formulas the same as for integral error functions arise from the properties of hypergeometric function :

$$\frac{\partial\Phi(a,\gamma,z)}{\partial z} = \frac{a}{\gamma}\Phi(a+1,\gamma,z) \qquad (3.2.29)$$

$$\Phi(a+1,\gamma+1,z) = \frac{\gamma}{z}[\Phi(a+1,\gamma,z) - \Phi(a,\gamma,z)]. \qquad (3.2.30) \quad [6]$$

3.3 The Gamma function and its properties

In mathematics, the gamma function (represented by the capital Greek alphabet letter Γ is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer:

$$\Gamma(n) = (n-1)!$$

The gamma function is defined for all complex numbers except the non-positive integers. For complex numbers with a positive real part, it is defined via a convergent improper integral:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

This integral function is extended by analytic continuation to all complex numbers except the non-positive integers (where the function has simple poles), yielding the meromorphic function we call the gamma function. It has no zeroes, so its reciprocal $1/?(z)$ is a holomorphic function. In fact the gamma function corresponds to the Mellin transform of the negative exponential function:

$$\Gamma(z) = \{\mathcal{M}e^{-x}\}(z)$$

The gamma function is a component in various probability-distribution functions, and as such it is applicable in the fields of probability and statistics, as well as combinatorics.[5]

Main definition The notation $\Gamma(z)$ is due to Legendre. If the real part of the complex number z is positive ($Re(z) > 0$), then the integral

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

converges absolutely, and is known as the Euler integral of the second kind (the Euler integral of the first kind defines the beta function). Using integration by parts, one sees that:

$$\Gamma(z+1) = \int_0^\infty x^z e^{-x}\, dx$$

$$= \left[-x^z e^{-x}\right]_0^\infty + \int_0^\infty z x^{z-1} e^{-x}\, dx$$

$$= \lim_{x\to\infty}(-x^z e^{-x}) - (0e^{-0}) + z\int_0^\infty x^{z-1} e^{-x}\, dx$$

Recognizing that as

$$x \to \infty, -x^z e^{-x} \to 0,$$

$$= z\int_0^\infty x^{z-1} e^{-x}\, dx = z\Gamma(z)$$

We can calculate $\Gamma(1)$:

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty$$

$$= \lim_{x\to\infty}(-e^{-x}) - (-e^{-0}) = 0 - (-1) = 1$$

Given that

$$\Gamma(1) = 1 \text{ and } \Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n) = 1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!,$$

for all positive integers n. This can be seen as an example of proof by induction. The identity

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},$$

can be used (or, yielding the same result, analytic continuation can be used) to uniquely extend the integral formulation for $\Gamma(z)$ to a meromorphic function defined for all complex numbers z, except integers less than or equal to zero.

It is this extended version that is commonly referred to as the gamma function.

Euler's definition as an infinite product

When seeking to approximate z! for a complex number z, it turns out that it is effective to first compute n! for some large integer n, then use that to approximate a value for (n+z)!, and then use the recursion relation m! = m (m?1)! backwards n times, to unwind it to an approximation for z!. Furthermore, this approximation is exact in the limit as n goes to infinity.

Specifically, for a fixed integer m, it is the case that

$$\lim_{n\to\infty} \frac{n!\,(n+1)^m}{(n+m)!} = 1\,.$$

and we can ask that the same formula is obeyed when the arbitrary integer m is replaced by an arbitrary complex number z

$$\lim_{n\to\infty} \frac{n!\,(n+1)^z}{(n+z)!} = 1\,.$$

Multiplying both sides by z! gives

$$z! = \lim_{n\to\infty} n!\frac{z!}{(n+z)!}(n+1)^z = \lim_{n\to\infty} \frac{1\cdots n}{(1+z)\cdots(n+z)}(n+1)^z = \prod_{n=1}^{\infty}\left[\frac{1}{1+\frac{z}{n}}\left(1+\frac{1}{n}\right)^z\right].$$

This infinite product formula converges for all complex numbers z except the negative integers, which fail because trying to use the recursion relation m! = m (m?1)! backwards through the value m = 0 involves a division by zero.

Similarly for the gamma function, the definition as an infinite product due to Euler is valid for all complex numbers z except the non-positive integers:

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}.$$

By this construction, the gamma function is the unique function that simultaneously satisfies

$$\Gamma(1) = 1, \Gamma(z+1) = z\Gamma(z)$$

for all complex numbers z except the non-positive integers, and

$$\lim_{n\to\infty} \frac{\Gamma(n+z)}{(n-1)!\,n^z} = 1 \text{ for all complex numbers } z.$$

IV. GIVEN PROBLEM AND ITS SOLUTION

4.1 Formulation of problem

Heat transfer in any coordinates can be described by the equation:

$$\frac{\partial U}{\partial t} = a^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial U}{\partial r}\right), \quad 0 < r < \alpha(t), \ t > 0 \quad (1.1.1)$$

With boundary and heat flux conditions

$$\left.\frac{\partial U}{\partial r}\right|_{r=0} = 0 \qquad (4.1.2)$$

$$\left.\frac{\partial U}{\partial r}\right|_{r=\alpha(t)} = f(t) = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot t^n \qquad (4.1.3)$$

$$\beta = \frac{\nu - 1}{2}, \qquad \alpha(t) = \sum_{n=1}^{\infty}\alpha_n t^n$$

4.2 Solution of Problem

We represent the solution in the form

$$\frac{\partial U}{\partial t} = a^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{\nu}{r}\cdot\frac{\partial U}{\partial r}\right), \quad 0 < r < \alpha(t), \ t > 0 \quad (4.2.1)$$

$$\left.\frac{\partial U}{\partial r}\right|_{r=0} = 0 \qquad (4.2.2)$$

$$\left.\frac{\partial U}{\partial r}\right|_{r=\alpha(t)} = f(t) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot t^n \qquad (4.2.3)$$

$$\beta = \frac{\nu-1}{2}, \qquad \alpha(t) = \sum_{n=1}^{\infty}\alpha_n t^n$$

Solution:

$$U(r,t) = \sum_{n=0}^{\infty} A_n Q_{n,\nu}$$

where $Q_{n,\nu} = \sum_{k=0}^{n} 2^{2k}\frac{n!\Gamma(\beta-1)\cdot r^{2n-2k}\cdot t^k}{k!\cdot(n-k)!\cdot\Gamma(\beta+1+n-k)} = \sum_{k=0}^{n} 2^{2k}\frac{n!\cdot\beta\cdot\Gamma(\beta)\cdot r^{2n-2k}\cdot t^k}{k!\cdot(n-k)!\cdot(\beta+n-k)\cdot\Gamma(\beta+n-k)} = \ldots =$

$$= \sum_{k=0}^{n} 2^{2k}\frac{n!\cdot\beta\cdot\Gamma(\beta)\cdot r^{2n-2k}\cdot t^k}{k!\cdot(n-k)!\cdot\beta\cdot\Gamma(\beta)\cdot(\beta-1)\cdot(\beta+2)\cdot\ldots\cdot(\beta+n-k)} = \sum_{k=0}^{n} 2^{2k}\frac{n!\cdot r^{2n-2k}\cdot t^k}{k!\cdot(n-k)!\cdot(\beta+1)\cdot(\beta+2)\cdot\ldots\cdot(\beta+n-k)}$$

$$Q_{0,\nu}(r,t) = 1$$

$$Q_{1,\nu}(r,t) = \left(2^0\cdot\frac{1!\cdot\Gamma(\beta-1)\cdot r^2}{0!\cdot 1!\cdot(\beta+1)\cdot\Gamma(\beta+1)} + 2^{2\cdot 1}\cdot\frac{1!\cdot\Gamma(\beta+1)\cdot t}{1!\cdot 0!\cdot\Gamma(\beta+1)}\right) = \frac{2r^2}{\nu+1} + 4t$$

$$Q_{2,\nu}(r,t) = \left(2^0\cdot\frac{2!\cdot\Gamma(\beta+1)\cdot r^4}{0!\cdot 2!\cdot(\beta+2)\cdot(\beta+1)\cdot\Gamma(\beta+1)} + 2^{2\cdot 1}\cdot\frac{2!\cdot\Gamma(\beta+1)\cdot r^2 t}{1!\cdot 1!\cdot(\beta+1)\cdot\Gamma(\beta+1)} + 2^{2\cdot 2}\cdot\frac{2!\cdot\Gamma(\beta+1)\cdot t^2}{2!\cdot 0!\cdot\Gamma(\beta+1)}\right) =$$

$$\frac{4r^4}{(\nu+1)\cdot(\nu+3)} + \frac{16r^2t}{\nu+1} + 16t^2$$

$$Q_{3,\nu}(r,t) = \frac{8r^6}{(\nu+1)\cdot(\nu+3)\cdot(\nu+5)} + \frac{3\cdot4r^4t\cdot4}{(\nu+1)\cdot(\nu+3)} + \frac{2\cdot3r^2t^2\cdot2^4}{\nu+1} + 64t^3$$

$$U(r,t) = \sum_{n=0}^{\infty} A_n Q_{n,\nu} = A_0\cdot1 + A_1\cdot\left(\frac{2r^2}{\nu+1} + 4t\right) +$$

$$+A_2\cdot\left(\frac{4r^4}{(\nu+1)\cdot(\nu+3)} + \frac{16r^2t}{\nu+1} + 16t^2\right) + A_3\cdot\left(\frac{8r^6}{(\nu+1)\cdot(\nu+3)\cdot(\nu+5)} + \frac{3\cdot4r^4t\cdot4}{(\nu+1)\cdot(\nu+3)} + \frac{2\cdot3r^2t^2\cdot2^4}{\nu+1} + 64t^3\right) +$$

$$\ldots + A_n\cdot\left(\frac{2^n\cdot r^{2n}}{(\nu+1)(\nu+3)\ldots(\nu+2n-1)} + \frac{2^{n-1}\cdot n\cdot r^{2n-2}\cdot t\cdot2^{2\cdot1}}{(\nu+1)(\nu+3)\ldots(\nu+2n-3)} + \ldots + \frac{n\cdot2^{2(n-1)}\cdot2^1\cdot r^2\cdot t^{n-1}}{\nu+1} + 2^{2n}\cdot t^n\right) + \ldots$$

$$\left.\frac{\partial U}{\partial r}\right|_{r=\alpha(t)} = \left[\sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \frac{n!2^{n+k}\cdot t^k\cdot r^{2n-2k}}{k!\cdot(n-k)!\cdot(\nu+1)\cdot(\nu+3)\cdot\ldots\cdot(\nu+2(n-k)-1)}\right]' =$$

$$= \left[\sum_{n=1}^{\infty} A_n \sum_{k=0}^{n} \frac{n!2^{n+k}\cdot(2n-2k)\cdot t^k\cdot(\alpha(t))^{2n-2k-1}}{k!\cdot(n-k)!\cdot(\nu+1)\cdot(\nu+3)\cdot\ldots\cdot(\nu+2(n-k)-1)}\right] = A_1\cdot\left[\frac{4\cdot\alpha(t)}{\nu+1}\right] + \left(\frac{2^n\cdot r^{2n}}{(\nu+1)(\nu+3)\ldots(\nu+2n-1)}\right.$$

$$\left. + \frac{2^{n-1}\cdot n\cdot r^{2n-2}\cdot t\cdot2^{2\cdot1}}{(\nu+1)(\nu+3)\ldots(\nu+2n-3)} + \ldots + \frac{n\cdot2^{2(n-1)}\cdot2^1\cdot r^2\cdot t^{n-1}}{\nu+1} + 2^{2n}\cdot t^n\right) + \ldots$$

$$A_2\cdot\left[\frac{2!\cdot2^2\cdot4(\alpha(t))^3}{2!\cdot(\nu+1)\cdot(\nu+3)} + \frac{2!\cdot2^3\cdot2(\alpha(t))\cdot t}{1!\cdot(\nu+1)}\right] + \ldots + A_n\cdot\left[\frac{2^n\cdot2n\cdot(\alpha(t))^{2n-1}}{(\nu+1)(\nu+3)\ldots(\nu+2n-1)} + \frac{n\cdot2^{n-1}\cdot2(n-1)\cdot(\alpha(t))^{2n-3}\cdot t\cdot2^{2\cdot1}}{(\nu+1)(\nu+3)\ldots(\nu+2n-3)} + \right.$$

$$\left.\ldots + \frac{n\cdot2^{2(n-1)}\cdot2^1\cdot2\cdot(\alpha(t))\cdot t^{n-1}}{\nu+1}\right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}t^n$$

$$\left.\frac{\partial U_1}{\partial t}\right|_{t=0} = A_1\frac{4\alpha}{\nu+1} = f'(0) \qquad \Rightarrow \qquad A_1 = \frac{f'(0)(\nu+1)}{4\alpha}$$

$$\left.\frac{\partial^2 U_1}{\partial t^2}\right|_{t=0} = A_2\frac{32*2!\alpha}{\nu+1} = f''(0) \qquad \Rightarrow \qquad A_2 = \frac{f''(0)(\nu+1)}{64\alpha}$$

$$\left.\frac{\partial^3 U_1}{\partial t^3}\right|_{t=0} = A_2\frac{16*3!\alpha^3}{(\nu+1)(\nu+3)} + A_3\frac{96*2*3!\alpha}{(\nu+1)} = f'''(0) \Rightarrow A_3 = \frac{f'''(0)(\nu+1)}{96*2*3!\alpha} - \frac{3f''(0)\alpha^2(\nu+1)}{96*2*3!}$$

$$\left.\frac{\partial^4 U_1}{\partial t^4}\right|_{t=0} = A_3\frac{3*4*4*\alpha^3 4!}{(\nu+1)(\nu+3)} + A_4\frac{4*2^8*4!\alpha}{(\nu+1)} = f^{(IV)}(0) \Rightarrow A_4 = \frac{f^{(IV)}(0)(\nu+1)}{4*2^8*4!\alpha} -$$

$$-\frac{\nu+1}{(\nu+3)4*2^8*4!}\left(f^{(III)}(0)\alpha + \frac{3}{2}f^{(II)}(0)\right)$$

$$\left.\frac{\partial^5 U_1}{\partial t^5}\right|_{t=0} = A_3\frac{8*6*\alpha^5*5!}{(\nu+1)(\nu+3)(\nu+5)} + A_4\frac{4*2^5*6*6*\alpha^5*5!}{(\nu+1)(\nu+3)(\nu+5)} + A_5\frac{5*2^8*2*2*\alpha 5!}{(\nu+1)} = f^{(V)}(0)$$

...

...

4.3 A Boundary Problems for the Generalized Heat Equation in the Domain with Moving Boundary

The solution of the problem

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial z^2} + \frac{c}{z}\frac{\partial\theta}{\partial z}\right), \quad 0 < z < \alpha(t),$$

$$\theta(0,t) = f(t), \qquad \theta(\alpha(t),t) = g(t)$$

where f(t) and g(t) are analytical functions, can be represented in the form of linear combination using degenerate hypergeometric function. We interested to find approximate solutions of this problem in the finite series.

$$\theta(z,t) = \sum_{n=0}^{m}(2a\sqrt{t})^n\left[A_n\Phi\left(-\frac{n}{2}, \frac{c+1}{2}, \frac{z^2}{4a^2t}\right) + B_n\Psi\left(-\frac{n}{2}, \frac{c-1}{2}, -\frac{z^2}{4a^2t}\right)\right]$$

from the first condition when z=0 we take

$$\sum_{n=0}^{m}(2a\sqrt{t})^n[A_n + B_n] = \sum_{n=0}^{m} f_n t^n$$

or in this form dividing to even and odd series

$$\sum_{n=0}^{m}(2a\sqrt{t})^{2n}[A_{2n} + B_{2n}] + \sum_{n=0}^{m}(2a\sqrt{t})^{2n+1}[A_{2n+1} + B_{2n+1}] = \sum_{n=0}^{m} f_n t^n$$

after compare both side by power of t

$$A_{2n} + B_{2n} = \frac{f_n}{(2a)^{2n}}$$

$$A_{2n-1} + B_{2n-1} = 0$$

from second condition at the $z = \alpha\sqrt{t}$

$$\sum_{n=0}^{m}(2a\sqrt{t})^n\left[A_n\Phi\left(-\frac{n}{2}, \frac{c-1}{2}, \frac{\alpha^2}{4a^2t}\right) + B_n\Psi\left(-\frac{n}{2}, \frac{c+1}{2}, -\frac{\alpha^2}{4a^2t}\right)\right] = g(t)$$

So if we again divide to even and odd part

$$\sum_{n=0}^{m}(2a\sqrt{t})^{2n}\left[A_{2n}\Phi\left(-n, \frac{c-1}{2}, \frac{\alpha^2}{4a^2t}\right) + B_{2n}\Psi\left(-n, \frac{n+1}{2}; -\frac{\alpha^2}{4a^2t}\right)\right] +$$

$$+\sum_{n=0}^{m}(2a\sqrt{t})^{2n+1}[A_{2n+1}\Phi(-\frac{2n-1}{2},\frac{v-1}{2},\frac{a^2}{4a^2t})-B_{2n+1}\Psi(-\frac{2n+1}{2},\frac{v-1}{2},-\frac{a^2}{4a^2t})]=$$

$$=\sum_{n=0}^{m}g_n t^n$$

$$(2a)^{2n}[A_{2n}\Phi(-n,\frac{v-1}{2},\frac{a^2}{4a^2t})+B_{2n}\Psi(-n,\frac{v+1}{2},-\frac{a^2}{4a^2t})]=g_n$$

$$[A_{2n+1}\Phi(-\frac{2n+1}{2},\frac{v+1}{2},\frac{a^2}{4a^2t})+B_{2n+1}\Psi(-\frac{2n-1}{2},\frac{v+1}{2},-\frac{a^2}{4a^2t})]=0$$

Now using these equation we make the system of equations

$$A_{2n-1}+B_{2n-1}=0$$

$$[A_{2n+1}\Phi(-\frac{2n+1}{2},\frac{v+1}{2},\frac{a^2}{4a^2t})+B_{2n+1}\Psi(-\frac{2n-1}{2},\frac{v+1}{2},-\frac{a^2}{4a^2t})]=0$$

Solution of this system only the trivial, when $A_{2n+1}=B_{2n+1}=0$

and the second system give the value of A_{2n}, B_{2n}

$$A_{2n}+B_{2n}=\frac{f_n}{(2a)^{2n}}$$

$$(2a)^{2n}[A_{2n}\Phi(-n,\frac{v-1}{2},\frac{a^2}{4a^2t})+B_{2n}\Psi(-n,\frac{v+1}{2},-\frac{a^2}{4a^2t})]=g_n$$

$$B_{2n}=\frac{1}{(2a)^{2n}}\frac{g_n-f_n\Phi(-n,\frac{v-1}{2},\frac{a^2}{4a^2t})}{\Psi(-n,\frac{v+1}{2},-\frac{a^2}{4a^2t})-\Phi(-n,\frac{v+1}{2},\frac{a^2}{4a^2t})}$$

$$A_{2n}=\frac{1}{(2a)^{2n}}\frac{f_n\Psi(-n,\frac{v-1}{2},-\frac{a^2}{4a^2t})-g_n}{\Psi(-n,\frac{v-1}{2},-\frac{a^2}{4a^2t})-\Phi(-n,\frac{v-1}{2},\frac{a^2}{4a^2t})}$$

And solution of given problem will be this form

$$\theta(z,t)=\sum_{n=0}^{m}(2a\sqrt{t})^n[A_n\Phi(-n,\frac{v+1}{2},\frac{z^2}{4a^2t})+B_n\Psi(-n,\frac{v-1}{2},-\frac{z^2}{4a^2t})]$$

We interested to find solution another problem problem half infinite domain

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial t^2} + \frac{v}{z}\frac{\partial\theta}{\partial z}\right), \quad \alpha(t) < z < \infty.$$

$$\theta(z,0) = f(z), \qquad \theta(\alpha\sqrt{t},t) = g(t), \qquad \theta(\infty,t) = 0$$

where f(z)and g(t) are analytical functions, can be represented in the form of linear combination using degenerate hypergeometric function. We interested to find approximate solutions of this problem in the finite series.

$$\theta(z,t) = \sum_{n=0}^{m}(2a\sqrt{t})^n[A_n\Phi(-\tfrac{n}{2},\tfrac{v+1}{2},\tfrac{z^2}{4a^2t}) + B_n\Psi(-\tfrac{n}{2},\tfrac{v-1}{2},-\tfrac{z^2}{4a^2t})]$$

from second condition at the $z = \alpha(t)$

$$\sum_{n=0}^{m}(2a\sqrt{t})^n[A_n\Phi(-\tfrac{n}{2},\tfrac{v-1}{2},\tfrac{\alpha^2}{4a^2t}) + B_n\Psi(-\tfrac{n}{2},\tfrac{v+1}{2},-\tfrac{\alpha^2}{4a^2t})] = g(t)$$

So if we again divide to even and odd part

$$\sum_{n=0}^{m}(2a\sqrt{t})^{2n}[A_{2n}\Phi(-n,\tfrac{v-1}{2},\tfrac{a^2}{4a^2t}) + B_{2n}\Psi(-n,\tfrac{v+1}{2},-\tfrac{\alpha^2}{4a^2t})]+$$

$$+\sum_{n=0}^{m}(2a\sqrt{t})^{2n-1}[A_{2n-1}\Phi(-\tfrac{2n-1}{2},\tfrac{v-1}{2},\tfrac{\alpha^2}{4a^2t}) + B_{2n+1}\Psi(-\tfrac{2n+1}{2},\tfrac{v-1}{2},-\tfrac{\alpha^2}{4a^2t})] =$$

$$= \sum_{n=0}^{m} g_n t^n$$

$$(2a)^{2n}[A_{2n}\Phi(-n,\tfrac{v-1}{2},\tfrac{\alpha^2}{4a^2t}) + B_{2n}\Psi(-n,\tfrac{v+1}{2},-\tfrac{\alpha^2}{4a^2t})] = g_n$$

$$[A_{2n+1}\Phi(-\tfrac{2n+1}{2},\tfrac{v+1}{2},\tfrac{\alpha^2}{4a^2t}) + B_{2n-1}\Psi(-\tfrac{2n-1}{2},\tfrac{v+1}{2},-\tfrac{\alpha^2}{4a^2t})] = 0$$

Using this properties apply to initial conditions

$$\lim_{\frac{z}{4a\sqrt{t}}\to\infty}\left(\frac{2a\sqrt{t}}{z}\right)^n\Psi(-\tfrac{n}{2},\tfrac{v+1}{2},-\tfrac{z^2}{4a^2t}) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v+1}{2}+\frac{n}{2})}$$

$$\sum_{n=0}^{m} B_n\frac{\Gamma(\frac{v-1}{2})}{\Gamma(\frac{v-1}{2}+\frac{n}{2})}z^n = \sum_{n=0}^{m}\frac{f^{(n)}(0)}{n!}z^n$$

where

$$B_n = \frac{f^{(n)}(0)}{n!}\frac{\Gamma(\frac{s+1}{2}+\frac{n}{2})}{\Gamma(\frac{s+1}{2})}$$

so B_n, A_n are founded.

V. APPLICATION

The solution of the problem for the equation

$$\frac{\partial U}{\partial t} = a^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\nu}{x} \cdot \frac{\partial U}{\partial x}\right).$$

in the domain D: $0 < x < \alpha t,\ t > 0$ with the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 0$$

$$u(\alpha t, t) = g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n$$

can be found in the form

$$u(x,t) = \sum_{n=0}^{\infty} A_n Q_n(x,t)$$

If $\nu \neq 0$, the first condition is necessary for the main given equation. By using the last representation, it is satisfied automatically. The coefficients A_n can be found by calculation.

It is very important for the theory of electrical contacts to find the temperature distribution in the system consisting of a liquid metal bridge and solid electrode. The temperature field inside bridge can be described by the equation with a variable cross-section

$$\frac{\partial \theta_1}{\partial t} = a_2^2\left(\frac{\partial^2 \theta_1}{\partial x^2} + \frac{\nu}{x}\frac{\partial \theta_1}{\partial x}\right), \quad -\alpha(t) < x < 0,\ t > 0$$

and inside the solid contact by the spherical equation

$$\frac{\partial \theta_2}{\partial t} = a_2^2\left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{2}{r}\frac{\partial \theta_2}{\partial r}\right); \quad r_0 < r < \infty,\ t > 0$$

The initial conditions are

$$\alpha(0) = 0,\ \theta_1(0,0) = \theta_m,\ \theta_2(r,0) = f(r),\ f(r_0) = \theta_m.$$

The boundary conditions are

$$\frac{\partial \theta_1(-r_0, t)}{\partial x} = 0$$

$$\theta_1(0, t) = \theta_2(r_0, t)$$

$$\lambda_1 \frac{\partial \theta_1(0, t)}{\partial x} = 2\lambda_2 \frac{\partial \theta_1(0, t)}{\partial x}$$

$$\theta_2(\infty, t) = 0$$

VI. CONCLUSION

The One Phase Stefan problem by new method, Integral Error Function, has been considered. Analytical solution has been found. Additionally, considered particular case, such as using "Matlab". Solution has been shown in graph.

Novelty of the thesis is that, Generalized Heat equation was reviewed from another side: using the Laguerre polynomials. Laquerre polynomials have obtained from Heat polynomials. At the same time, Heat polynomials interconnected. One of the difference of the boundary problem is that, it has different radiuses at each point.

Additionally, we have considered Generalized Heat equation with moving boundary according to its domain. By giving another initial and boundary conditions, we have solved the equation.

The application of the obtained results for the calculation of the electrical arc heat flux at the contact opening is presented.

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