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Fujita hypothesis and birational models in algebraic geometry

THESIS

Presented in Partial Fulfillment for the
Degree of Master of Science in Science in Mathematics
(degree code: 6M060100)
Department of Mathematics and Natural sciences
Faculty of Engineering and Natural Sciences

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Kaskelen, 2019

Abstract

The aim of my research is to study Fujita hypotheses, which states that if X is a smooth projektive varyety of dimension n then:

(i) Assume that X is a minimal (i.e., K_X is nef) varyety of general type (i.e., K_X is big $\Longleftrightarrow K_X^n > 0$). Then the linear system $|mK_X|$ is free for $m \geq n + 2$;

(ii) Let A be an ample invertible sheaf on X. Then the linear system $|mK_X + (n+1)A|$ is free, and $|mK_X + (n+2)A|$ is very ample on X.

For surfaces Fujita hypotheses was proved by Igor Reider.

In my work I generalize result for X not be a smooth projektive varyety but F-rational. So if Fujita hypotheses is true for a smooth projektive varyety then it is true for more general case F-rational. I use methods of commutative algebra and definition of tight closure.

Аңдатпа

Менің зерттеулерімнің мақсаты X тегіс n - өлшемді проективті көпбейне болған кездегі Фуджита гипотезасын зерттеу. Онда:

(i) X жалпы түрдің минималды көпбейнесі деп алайық. Онда $|mK_X|$ сызықты жүйесі $m \geq n+2$ үшін бос.

(ii) A толық оралымды X-тағы бума. Онда $|mK_X + (n+1)A|$ сызықты жүйесі бос. және $|mK_X + (n+2)A|$ X-та толық жеткілікті.

Фуджитаның кейбір жағдайларын Игор Райдер дәлелдеді.

Мен жұмысымда $X-$ тегіс емес проекивті көпмүше бірак $F-$ рационалды үшін нәтижені көрсетемін.

Аннотация

Цель моего исследования - изучить гипотезы Фудзиты, в которых говорится, что если X - гладкое проективное многообразие размерности n, то:

(i) Предположим, что X - минимальное многообразие общего типа (т. е. K_X большое $\Longleftrightarrow K_X^n > 0$). Тогда линейная система $|mK_X|$ свободен для $m \geq n+2$;

(ii) Пусть A - вполне обратимый пучок на X. Тогда линейная система $|mK_X + (n+1)A|$ свободен, и $|mK_X + (n+2)A|$ полностью достаточно на X.

Поверхностей гипотезы Фуджита подтвердил Игорь Рейдер.

В моей работе я обобщаю результат для X не гладкого проективного многообразия, но -рационального. Таким образом, если гипотезы Фуджиты верны для гладкого проективного многообразия, то это верно для более общего случая -рациональных. Я использую методы коммутативной алгебры и определения плотного замыкания.

Acknowledgements

Thanks to my thesis supervisor for constant support and useful discussion. Thanks my friends and family for their great support

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1. Introduction

My diploma thesis is dedicated to the branch of algebraic geometry, concerned with properties of Divizors. The Divizor is one of main tool of algebraic geometry in order to study properties of affine and projektive manifolds. Also I have studied the basics of commutative algebra such as Zarisky topology and affine scheme. Starting from the theory of algebraic curves, the main problem of the branch of algebraic geometry that studies maps $X \to \mathbb{P}^n$ is to find conditions under which the linear system $|D|$ defining this map does not have base points or is very ample. Multiply canonical linear systems $|mK_X|$ for surfaces of general type were studied by Kodaira and Bombieri.

The aim of my research is to study Fujita hypotheses, which states that if X is a smooth projektive varyety of dimension n then:

(i) Assume that X is a minimal (i.e., K_X is nef) varyety of general type (i.e., K_X is big $\iff K_X^n > 0$). Then the linear system $|mK_X|$ is free for $m \geq n+2$;

(ii) Let A be an ample invertible sheaf on X. Then the linear system $|mK_X + (n+1)A|$ is free, and $|mK_X + (n+2)A|$ is very ample on X.

For surfaces Fujita hypotheses was proved by Igor Reider in 1988.

For 3 dimensional algebraic manifolds Lazarsfeld and Ein in 1993 proved the 1-st part of the Fujita conjecture, i.e. that $m \geq 1$ mean global generation.

In my work I generalize result for X not be a smooth projektive varyety but F-rational. Also I generalize results of Reider and Iskovskih of for positive simple characteristic. So If Fujita hypotheses is true for a smooth projektive varyety then it is true for more general case F-rational.

I use methods of commutative algebra and definition of tight closure, defined by by Melvin Hochster and Craig Huneke (1988, 1990).

2. Preliminaries

Algebraic variaty - first of the main object for the study of algebraic geometry. The basic definition of this variaty is the set of solutions of these algebraic equations in real or complex numbers. New definitions generalize it in various ways, but try to preserve the geometric intuition corresponding to this definition [1].

Any authors may have different definitions of this variaty: any authors [2] include the property of irreducibility in the definition (this means that a variaty cannot be a union of smaller varieties), while any [3] distinguish irreducible and "common" varieties. In this paper we will adhere to the first agreement, and we will call the solution sets of systems of equations that are not irreducible, algebraic sets.

The concept of an algebraic variaty has any similarities with the concept of a smooth variaty. The difference is that algebraic varieties, unlike smooth varieties, can have singular points. The neighborhood of a non-singular point of a real algebraic variaty is isomorphic to assmooth variaty.

Hilbert's theorem on zeros of algebra proved around 1800 established a connection between geometry and algebra, showing that a given polynomial in one variable (an object from an algebra) is uniquely determined by its complex roots, that is, on a complex plane by a finite set of points (an object from geometries). The main theorem of algebra, generalizing this result, established the main correspondence with algebraic varieties and ideals of the polynomial ring. Using the main theorem of algebra related results and the theorem itself, mathematicians established a correspondence between questions of the theory of rings and questions about algebraic varieties; the use of such correspondences is usually a distinguishing feature of algebraic geometry

Algebraic diversity, the main object of study in algebraic geometry, which was initially defined as a set of points in n-dimensional space, whose coordinates $x_1, \ldots, x_n$ are solutions of the system of equations

$$F_1(x_1,\ldots,x_n)=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2.1)$$
$$F_n(x_1,\ldots,x_n)=0$$

$F_1 \ldots\ldots F_m$ – polynomials from $x_1,\ldots,x_n$.
Each algebraic variaty has a certain dimension, which is the number of independent parameters defining a point on the variaty. Examples of algebraic curves are conic sections.
Later, affine, projective, and given abstract algebraic varieties began to differ. For each algebraic variaty, its field of definition k is fixed, which is also called the field of constants or the main field. An affine algebraic variaty is given by a system of algebraic equations of the form (4.18) in an affine space A_k^n over a field k with coordinates $x_1,\ldots,x_n$. Similarly, a projective algebraic manifold is eliminated by a system of homogeneous algebraic equations in the projective space P_k^n with homogeneous coordinates $y_0, y_1,\ldots,y_n$. Abstractly defined algebraic varieties are glued together from affine, just as ordinary differential varieties are glued together from Euclidean balls. Not every abstract algebraic variaty can be embedded into projective space.
On each algebraic variaty X, the Zariski topology is introduced, in which all submanifolds (including X itself and the empty set) are closed subsets. This topology is not strongly separable: for example, on an affine line A_k^1 over an infinite field k, any two open non-empty subsets intersect.
The Zariski topology allows us to define on X a bundle of local rings 0_X, called structural (see Puchkov theory). The layer 0_X of the sheaf O_x at the point $x=(x_1,\ldots,x_n)\in X$ consists of germs of regular functions in x. Each germ is (ambiguously) represented in any neighborhood of x by a rational function of the form
$$\frac{P(x_1,\ldots,x_n)}{Q(x_1,\ldots,x_n)}$$
where P and Q are polynomials and $Q(x)\neq 0$. The ring of global sections $A(X)$ of 0_X is called the coordinate ring on X – this is a ring of functions that are regular on the whole variaty X.
For an algebraic variaty, two types of mappings are considered: regular (usually called morphisms) and rational. Regular mappings are locally defined in coordinates by polynomials, and rational mappings are defined by rational functions.

i.e., by relations of polynomials. The latter may not be everywhere defined. If a rational map has the opposite, it is called birational.

The dimension of the (irreducible) algebraic variaty X is defined topologically as the maximum length of a chain of different nested non-empty closed subsets in X or algebraically as the maximum number of algebraically independent rational functions on X. Both definitions give the same number. Algebraic variaty of dimension 1 - algebraic curves, dimension 2 - algebraic surfaces. A hypersurface in an n - dimensional affine or projective space is given by one equation and has the dimension $n-1$.

The main difference between a projective algebraic variaty and an affine algebraic variaty is its completeness, which is an algebraic analogue of the topological concept of compactness. Every projective variaty over a field of complex numbers C is compact, and an affine variaty of dimension greater than 0 is not compact. There are complete algebraic varieties that are not isomorphic to projective. On complete algebraic varieties everywhere, only constants are regular functions.

The most important for an algebraic variaty is the concept of non-singularity (smoothness). It is local and is defined for each point. The point $x \in X$ in the definition of system (4.18) is called nonsingular (smooth) if the rank of the matrix (F_i/x_j) is maximum at this point. This condition corresponds to the condition for the existence of an implicit function in the analysis. An algebraic variaty is called non-singular (smooth) if all its points are non-singular, otherwise it is special. Generalizations of an algebraic variaty are schemes and algebraic spaces.

Students are supposed to be familiar with the definitions of the ring and module over the ring. All rings will be associative, commutative, with a unit. Recall that the ideal I in ring A is a submodule A as a module over itself, that is, the subgroup A in addition, is stable with respect to multiplication by all ring elements. The factor module A/I then has the structure of a ring (i.e., multiplication by the rule $(a+I)(b+I) = ab+I$ is correctly defined) and is called a factor ring A by the ideal I. The ideal generated by the elements $a_1, \ldots, a_k$, denoted by $(a_1, \ldots, a_k)$. An ideal that does not coincide with the whole ring is called simple if $xy \in I$ implies $x \in I$ or $y \in I$, and maximal if it is not contained in any other own ideal. The domain of the main ideals is the integral ring, where any ideal is generated by one element.

Algebraic geometry section of mathematics, studying geometric objects associated with commutative rings: algebraic varieties and their various generalizations (algebraic spaces. schemes. etc.).In the "naive" formulation. the subject of A. g. Is the study of solution of algebraic of equation. Geometric intuition appears when all the "solution set" is identified with the "set of points in a certain coordinate space". If the situation has a size of two or three. the visibility of the situation is beyond doubt: however. geometric language is used under more general circumstances. This language suggests problems. constructions and types of reasoning. which are hardly natural from the point of view of pure algebra. In turn. algebra delivers powerful apparatus and a flexible, equally adapted to turn plausible arguments into evidence and to formulate them in the most natural and general form.

In algebraic geometries over the field of complex numbers. every algebraic the variaty is at the same time complex-analytical. differentiable and topological. space in the usual Hausdorf topology. This circumstance allows you to enter a number of classic. structures delivering such invariants to algebraic varieties. which are only with great difficulty or not at all possible to obtain purely algebraic. means. The concepts and results of Algebraic geometry are intensively used in number theory (trigonometric sum estimates. Diophantine equations). differential topology (differentiable structures and singularities). group theories (simple finite groups related to Lie groups. algebraic groups), differential theory equations (K-theory and the index of elliptic operators). the theory of complex spaces, category theory (then-poses. abelian categories). functional analysis (representation theory). In turn, algebraic geometry uses the ideas and methods of these disciplines.

3. Basics of algebraic geometry

3.1 Spectrum of a commutative ring

Each point $\overline{a} = (a_1, \ldots, a_n)$ of the coordinate space $\mathbb{C}^n$ corresponds to an ideal $I_{\overline{a}} = (x_1 - a_1, \ldots, x_n - a_n)$ in the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ consisting of polynomials equal to zero in $\overline{a}$. In general, to every point x (a topological space of varieties) X corresponds to an ideal I_x in the ring of functions (uninterrupted, smooth) on X, formed by all the functions that are junction to zero in x. These ideals are maximum. Therefore, it is important to study the set of maximal ideals of a given ring — this allows one to reconstruct (in a certain sense) the original set from a function ring. But for various reasons, it is more fruitful to study a lot of simple ideals.

Definition 1. The spectrum of a commutative ring A is the set of its simple ideals other than A. Designation: $Spec(A)$

The spectrum of a ring in mathematics is the set of all simple ideals of a given commutative ring. The spectrum is usually supplied with the Zariski topology and a bundle of commutative rings, which makes it locally ringed by space. The spectrum of the ring R (hereinafter, the word "ring" means "commutative ring with unit") is denoted by $Spec(R)$.

Definition 2. A commutative ring is a ring in which the multiplication operation is commutative (usually, its associativity and the existence of a unit are also implied). The properties of commutative rings are studied by commutative algebra.

Ideals and factor rings - The internal structure of a commutative ring is determined by the structure of its ideals, that is, non-empty subsets that are closed with respect to addition, as well as multiplication by an arbitrary element of the ring. Given a subset $F = \{f_j\}_{j \in J}$ of a commutative ring R, we can construct the smallest ideal containing this subset. Namely, this space of finite linear combinations of the form

$$r_1 f_1 + \cdots + r_n f_n$$

An ideal generated by a single element is called prime. A ring in which all ideals are principal is called a ring of principal ideals, two important examples of such rings are $\mathbb{Z}$ and a ring of polynomials over the field $k[x]$. Any ring has at least two ideals - a zero ideal and the ring itself. An ideal that is not contained in another improper (not coinciding with the ring itself) ideal is called maximal. It follows from Zorn's lemma that there exists at least one maximal ideal in any ring.

The definition of an ideal is constructed in such a way that allows you to "divide" the ring into it, that is, there is a factor ring R/I: this is a set of adjacent classes in I with operations

$(a + b) + (b + I) = (a + b) + I$ and $(a + I)(b + I) = ab + I$

These operations are correctly defined, for example, $(a + I)(b + I) = ab + aI + Ib + |*| = ab + I$, since aI belongs to I, etc. It is clear from this why the definition of an ideal such this.

Ring localization - is an operation, in a certain sense, opposite to taking a factor: in a factor ring, elements of a certain subset turn into zero, whereas in a localization, elements of a certain set become reversible. Namely, if S is a subset of R that is closed with respect to multiplication, then localization by S, denoted as $S^{-1}R$, consists of formal symbols of the form

$\frac{r}{s}$, where $r \in R$, $s \in S$

with the rule of reducing the numerator and denominator, similar to the usual rule (but not coinciding with it). The operations of addition and multiplication on such "fractions" are defined in the usual way.

In this language, Q is the localization of Z over the set of nonzero integers. The same operation can be performed with any integral ring at the site Z: localization $(R\{0\})^{-1}R$ is called the field of quotients of ring R. If S consists of all degrees of a fixed element f, the localization is denoted as R_f.

A particularly important type of ideals are simple ideals, often denoted by the letter p. By definition, a simple ideal is an improper ideal, such that if it contains the product of two elements, then at least one of these elements is contained in

it. Equivalent definition - the factor ring R/p is holistic. Another equivalent definition is that the complement $R\backslash p$ is closed with respect to multiplication. Localization of $(R\backslash p)^{-1}R$ is important enough to have its own designation: R_p. This ring has only one maximal ideal: pR_p. Such rings are called local.
Simple ideals are a key element of a geometric description of a ring, using the spectrum of $SpecR$. As a set, $SpecR$ consists of simple ideals. If R is a field, there is only one simple ideal (zero) in it, so the spectrum of the field is a point. Another example - $SpecZ$ contains one point for the zero ideal and one for each prime number p. The spectrum is equipped with the Zariski topology, in which open sets are sets of the form $D(f) = \{p \in SpecR,\ f \notin p\}$, where f is an arbitrary element of the ring. This topology differs from the usual examples of topologies from the analysis: for example, the closure of a point corresponding to a zero ideal is always the whole spectrum.
Spectrum definition is basic for commutative algebra and algebraic geometry. In algebraic geometry, the spectrum is supplied with a pencil $\mathcal{O}$. The pair "space and a pencil on it" is called an affine scheme. An affine scheme can restore the original ring by using the global section functor. Moreover, this correspondence is functorial: it associates with each homomorphism of the rings $f : R \to S$ a continuous mapping in the opposite direction:

$$SpecS \to SpecR,\ q \longmapsto f^{-1}(q)$$

Thus, the categories of affine schemes and commutative rings are equivalent. Consequently, many definitions applied to rings and their homomorphisms emerge from geometric intuition. Affine schemes are local data for schemes (much like the spaces R^n are local data of varieties), which are the main object of study of algebraic geometry.
As usual in algebra, a homomorphism is a map between algebraic objects, preserving their structure. In particular, the homomorphism of (commutative) rings with unity is
a map $f : R \to S$, such that

$$f(a+b) = f(a) + f(b),\ f(ab) = f(a) f(b) \text{ and } f(1) = 1$$

In this situation, S is also an R-algebra: indeed, elements of S can be multiplied by elements of R by the rule

$$s \circ s := f(r) \circ s$$

The kernel and image of the homomorphism f are the sets $\ker(f) = \{r \in R.\ f(r) = 0\}$ and $im(f) = f(R) = \{f(r).\ r \in R\}$. The kernel is an ideal in R. and the image is a subring of S.

The Krull dimension (or simply the dimension) is a way to measure the "size" of a ring. Namely. this is the maximum length n of a chain of simple ideals of the form

$$p_0 \subseteq \cdots \subseteq p_n.$$

For example. a field has dimension 0. because it has only one ideal. zero. The dimension of integers is one: a single chain of simple ideals has the form

$$0 = p_0 \subsetneq p\mathbb{Z} = p_1$$

Where p – prime number. A local ring with a maximal ideal m is called regular if its dimension is equal to the dimension m/m^2 as a vector space over R/m.

The Zariski topology on the affine space $\mathbb{A}^n$ over the field K is the topology structure. whose closed subsets are exactly the algebraic sets of the given space. Algebraic sets are sets of the form.

$$V(S) = \{x \in \mathbb{A}^n | \forall f \in S : f(x) = 0\}$$

where S is an arbitrary set of polynomials of n variables over a field K. The following identities are easily verified:

$$V(\emptyset) = \mathbb{A}^n.\ V(1) = \emptyset$$

$$V(S) = V((S))$$

ideal in a ring of polynomials generated by elements of S. And for any two ideal I and J.

$$V(I) \cup V(J) = V(IJ)$$

$$V(I) \cap V(J) = V(I + J)$$

Since the ring of polynomials over the field is Noetherian. the intersection of an infinite family of sets of the form $V(I)$ will be equal to the intersection of its finite subfamily and have the form $V(I)$. Since finite unions and arbitrary intersections of algebraic sets. and also $\mathbb{A}^n$ and the empty set are algebraic. algebraic sets are indeed closed sets of some topology (equivalently. additions to them, denoted by $D(S)$. are open sets topology).

If M is an affine algebraic subset of the affine space $\mathbb{A}^n$. then the Zariski topology

on it is the induced topology.
The elements of the projective space $\mathbb{P}^n$ are the equivalence classes of the elements $\mathbb{A}^{n+1}$ with respect to proportionality with respect to a scalar multiplication from K. Therefore, the elements of the polynomial ring are $k[x_0, \ldots, x_n]$ are not functions on $\mathbb{P}^n$, since one point has many equivalent representations that correspond to different values of the polynomial. However, for homogeneous polynomials, the condition of equality to zero at this point is correctly defined, since multiplication by a scalar "passes through" the application of the polynomial. Therefore, if S is a set of homogeneous polynomials, the definition

$$V(S) = \{x \in \mathbb{P}^n | f(x) = 0, \ \forall f \in S\}$$

Similarly, it is verified that this family of sets is a family of closed sets of a certain topology: you only need to replace the word "ideal" with "homogeneous ideal". A topology on an arbitrary projective submanifold is defined as an induced topology.
A useful feature of the Zariski topology is the existence of a fairly simple base of this topology. Namely, the base of the topology is open sets of the form $D(f)$, which are an addition to the set of zeros of the polynomial f (respectively, for projective varieties - the homogeneous polynomial f).
Any affine or projective variety is compact; also a compact is any open subset of a variety. Moreover, any algebraic variety is a Noetherian topological space.
On the other hand, an algebraic variety is not a Hausdorff space (if K is not a finite field). Since any point of an algebraic variety is closed.
The modern definition is based on the concept of the spectrum of a ring. Let some commutative ring A with unity be given. The spectrum of the ring $SpecA$ is the set of its all simple ideals, and these ideals themselves are points of the spectrum. The Zariski topology is introduced as follows - the sets of all simple ideals containing some set E or, equivalently, the ideal I generated by this set, are considered to be closed spectrum sets.

$$V(I) = \{P \in Spec(A) | I \subseteq P\}$$

It is easy to check all the axioms. For example, the fact that the union of two closed sets is closed follows from a chain of obvious inclusions:
$V(a \cap b) \subseteq V(ab) \subseteq V(a) \cup V(b) \subseteq V(a \cap b)$, is here $V(a) \cup V(b) = V(a \cap b)$
With the previously introduced topology on affine space, the Zariski topology on the spectrum is associated as follows. We define the map $\mathbb{A}^n \to SpecK[x_1, \ldots, x_n]$.

which associates with the point p the maximal ideal m_p, consisting of polynomials, equal to zero at this point (it is maximal, since the quotient ring on it is the field K). Obviously, different ideals correspond to different points. Moreover, the Hilbert theorem on zeros states that all maximal ideals of a polynomial ring have this form, that is: the map $x \longmapsto m_x$ is bijective. Moreover, this map is a homeomorphism $\mathbb{A}^n$ onto a subset $SpecK[x_1, \ldots, x_n]$, corresponding to maximal ideals (the set of maximal ideals of the ring A with induced the Zariski topology is called the maximal spectrum and is usually denoted by $SpecA$). It is enough to prove that this mapping induces a bijection between closed subsets $\mathbb{A}^n$ and closed subsets $SpecK[x_1, \ldots, x_n]$, but it is almost obvious: the maximal ideals containing ideal (S) are exactly the common zeros of all polynomials from S.
Thus, Grothendieck's innovation was to consider not only the maximal ideals of the ring, but also all the simple ideals. In the case of a polynomial ring over an algebraically closed field, this means that a certain number of "common points" is added to the space $\mathbb{A}^n$ (one point for each irreducible affine subvariety). In the general case (that is, when considering all possible commutative rings), this endows $Spec$ with functorial properties: to each homomorphism of the rings $A \to B$ there corresponds a continuous mapping $SpecB \to SpecA$. For a simple spectrum, the construction of this homomorphism is trivial - the pre-image of a simple ideal is taken, for a maximal it does not work this way, since the pre-image of a maximal ideal is not necessarily maximal.
The first way to introduce the *Zariski topology* on the spectrum of a ring is to indicate the base of the topology. The bases are the spectral subsets of the form $D_f = \{p \in SpecR : f \notin p\}$, where f is an arbitrary element of the ring R. The following statements are easily verified:

$$D_1 = SpecR$$

$$D_f \cap D_g = D_{fg}$$

From these formulas it follows that the family of all subsets of the form D_f is a spectrum covering that is closed with respect to intersections, that is, is the basis of some topology.
The spectrum of a ring, as a rule, is not a Hausdorff space. On the other hand, the spectrum of any ring satisfies the separation axiom T_0 and is compact.
To prove the compactness, it suffices to check that from the coverage of the

elements of the base one can choose the final subcovering. If the set system $D_f : f \in A$ is a spectrum coverage, this means that the ideal of the ring R, generated by the set A, contains one. That is, equality is true:
$1 = k_1 a_1 + \cdots + k_n a_n$, in which a_i are elements of the set A, and k_i are some elements of the ring R. But then $\{D_{f_1}, D_{f_2}, \ldots, D_{f_n}\}$ is the desired finite subcovering of the spectrum. The compactness of the sets D_f is proved similarly. (It should be noted that in the absence of Hausdorflity, a compact subset does not have to be closed!)
To prove the *equivalence of definitions through the base of topology* and through closed sets, it suffices to check the formulas:

$$V(a)^c = \cup_{f \in a} D_f$$

$$D_f = V((f))^c$$

Where V^c denotes the complement of the set V, and (f) is the ideal generated by the element f.
The structural beam on the spectrum is defined as follows: to each open set D_f from the base, the localization of the ring R is mapped by the multiplicative system $\{1, f, f^2, f^3, \ldots\}$. The elements of this localization are formal fractions of the form p/q, such that q is a power of f. Accordingly, the open set $\cup_i D_{f_i}$ is matched by localization on a multiplicative system generated by f_i.
The same open set can be represented as $\cup_i D_{f_i}$ in many ways, but it can be shown that the localization of the ring does not depend on the choice of such a representation and also verify that all other properties of the beam are satisfied.
In the case when R is an integral ring with a field of quotients K, the structural beam can be described more specifically. An element $f/g \in K$ is called regular at a point $p \in SpecR$, if it can be represented as a fraction f/g whose denominator does not belong mathfrak p. Accordingly, the open set U is associated with a set of elements K, which are regular at each point of U; one can verify that this set is closed with respect to addition and multiplication, that is, it forms a ring. The construction of restriction mappings in this case is also more obvious: if $U' \subset U$, then the element of the field of quotients regular at each point U is regular at each point U'.
The layer of the resulting bundle O at the point p coincides with the localization R_p of the ring R by the simple ideal p, this ring is local. Therefore, the spectrum

of the ring is indeed a locally ringed space.

Each homomorphism of the rings $\varphi : A \to B$ corresponds to a continuous mapping of the spectra (in the opposite direction) $\varphi^* : SpecB \to SpecA$. Indeed, the preimage the prime ideal $p \in B$ under the action of φ is a prime ideal. In order to prove the continuity of this mapping, it suffices to prove that the preimage of a closed set is closed. This follows from equality

$(\varphi^*)^{-1}(V(a)) = V(\varphi(a))$, where a is an arbitrary ideal of A.

It follows from this that $Spec$ is a contravariant functor from the category of commutative rings to the category of topological spaces. Moreover, the φ map for each $p \in B$ induces a homomorphism of local rings

$$\mathcal{O}_{f^{-1}(p)} \to \mathcal{O}_p$$

Consequently, $Spec$ defines a contravariant functor in the category of locally ringed spaces. The image of this functor is exactly affine schemes, therefore the category of commutative rings (contravariantly) is equivalent to the category of affine schemes.

In algebraic geometry, algebraic varieties are studied, that is, subsets of the space K^n (where K is an algebraically closed field) defined as common zeros of a certain set of polynomials in n variables. If A is such an algebraic variety, consider the commutative ring of polynomial functions $A \to K$. Then the maximal ideals of the ring R correspond to the points of the variety A, and the simple ideals all the irreducible subvarieties A (the variety is called irreducible if it cannot be represented as a union of two smaller varieties). Moreover, the closure of a subvariety consists of all its points and subvarieties. Moreover, the bundle defined above on the spectrum actually coincides with the bundle of rational functions on an algebraic variety A.

3.2 Sheaves

A *sheave* is a structure used to establish relationships between local and global data. Bundles play a significant role in topology, differential geometry and algebraic geometry, but they are also used in number theory, analysis and category theory.

Roughly speaking, the sheaf F on the topological space X is given by two types of data with two additional properties.

The first part of the data is enclosed in a mapping that associates with each open subset U of a space X a certain (abstract) set $F(U)$. It is possible to require, in addition, that a certain structure be given on this set, but for now we restrict ourselves to the fact that it is just a set.

The second part of the data consists in the fact that for each pair of open sets $V \subset U$ a certain $\rho_{VU} : F(U) \to F(V)$ map is fixed, which is called the restriction. (It acts like a narrowing operation on the area V of functions defined on U.)

The first step in defining a beam concept is to define the concept of a pre-sheaf, which covers the data spaces associated with each open subset of the topological space, and the operation to restrict this data from larger subsets to smaller ones. At the second step, additional restrictions are imposed - the requirements of the satisfiability of the axioms of normalization and gluing. The pre-sheaf satisfying these requirements is the bundle.

Let X be a topological space, and C a category. Over the space X, a pre-sheaf F is given with values in the category C, if:

1. Each open subset U in X is associated with $F(U)$ - an object of category C.

2. Each inclusion $V \subset U$ of open sets is associated with a morphism of objects of category C: $\rho_{VU} : F(U) \to F(V)$

These morphisms are called restriction morphisms. The combination of these morphisms must satisfy the following conditions:

1. For any open set, $U\rho_{UU}$ is the identity morphism of the object $F(U)$.

2. For each double inclusion $W \subset V \subset U$ the equality is true

3.

$$\rho_{WV} \circ \rho_{VU} = \rho_{WU}$$

The last condition means that it should be indifferent, we restrict the data from the area U to the area W directly, or in two stages - with a preliminary restriction to V, and from it already to W.

A very compact definition of a prepouch is obtained in terms of category theory. First, the category $O(X)$ of open sets of X is defined, whose objects are open subsets of X, and the set of morphisms of the object V of this category to the object U if V is a subset of U consists of a single morphism - the display of the inclusion V in U, and empty otherwise. Then a pre-sheaf over the space X with values in the category C is any contravariant functor F from the category $O(X)$ to the category C. Such a definition pre-leaves admits a further generalization when considering functors in C not necessarily from a category of the type $O(X)$ If the pre-sheaf X with values in the category F is specified over the space C, and U is an open subset of X, the object $F(U)$ is called the pre-sheaf space F of U over the set U. If C is a specific category, then each element of the set $F(U)$ is called of a sheaf F over U, by analogy with sections of stratified spaces and the etale space of a sheaf. Section over X is called a global section. The constraint $\rho_{VU}(s)$ of the section s is usually denoted as $s|_V$. $F(U)$ is also often denoted as $\Gamma(U, F)$, especially in the context of the cohomology theory of sheaves, in which the domain U is fixed and the pencil F is variable.

A bundle is a pre-sheaf in which 2 axioms are fulfilled:

1. $F(\emptyset)$ is a terminal object of category C. Of course, for the axiom to make sense, it is necessary for category C to have a terminal object. In practice, this is usually the case. However, the more important axiom is the gluing axiom. Recall that in the examples disassembled above, this axiom required that the data set (beam sections), agreed at the intersections of their domains, always allow (moreover) uniquely their gluing - the section over the union of open sets, over which this section is defined as partially. For simplicity, we formulate the gluing axiom in the case when C is a specific category. For the general case, see the "gluing axiom" article.

Let U_i be the set of open sets of the space X, and U be their union. Suppose each of them is given a section $s_i \in F(U_i)$ of a bundle F. A set of these sections is called consistent (compatible) if for all i and j

$$\rho_{U_i \cap U_j, U_i}(s_i) = \rho_{U_i \cap U_j, U_j}(s_j)$$

1. every set of matched sections s_i defines a unique section $s \in F(U)$, such that $s_i = \rho_{U_i, U} s$ for each i.

The section s is called the gluing of the sections s_i, since it is, as it were, glued together from smaller sections. In the examples given above, some functions corresponded to the beam sections. In such cases, the gluing axiom starts from the functions f_i, which coincide at the intersections $U_i \cap U_j$, and asserts the existence of a single function f that simultaneously extends all the functions f_i to the set U, - just , which was shown in those examples to prove that a bundle was actually presented in them.

Often the gluing axiom is divided into two parts - the axiom of existence and the axiom of uniqueness. Pre-shells that satisfy only the uniqueness axiom are called separable pre-shells.

4. Fujita problem

4.1 Reidesr's theorem and Divizors

1. Divisor function. A polynomial in one variable is uniquely determined to within a constant multiplier if one specifies its roots and their multiplicity, i.e., a set of points $x_1, \ldots, x_r \in \mathbb{A}^1$ with multiplicities $l_1, \ldots, l_r$. The rational function $\varphi(x) = \frac{f(x)}{g(x)}$, $f, g \in k[X]$ is dictated by the zeros of the polynomials f and g that is, that is, by the focuses at which it transforms into 0 or isn't standard. We also take with the minus sign of their multiplicities. To recognize the trenches of g from those of f we take their multiplicities with the sign minus. Thus, the function φ is given by anypoint $x_1, \ldots, x_r$ with arbitrary any integer multiplicities $l_1, \ldots, l_r$. Presently we set ourselves the objective to also set a levelheaded capacity on a subjective logarithmic varyety.

We will proceed from the fact that, according to the intersection dimensionality theorem, the set of points at which a regular function turns to zero, it formas submanifolds of codimension 1. Therefore, the object that we will associate with functions is a set of irreducible submanifolds of codimension 1 with the multiplicities assigned to them. We will give multiplicities integers both positive and negative.

De?nition. A collection of irreducible closed submanifolds $C_1, \ldots, C_r$ of codimension 1 in an irreducible varyety X with preassigned integral multiplicities $l_1, \ldots, l_r$ is called a *Divizor*.

A divisior D is written in the forma

$$D = l_1 C_1 + \cdots + l_r C_r \tag{4.1}$$

If if all the $l_i > 0$, then we write $D > 0$ and all the $l_i = 0$, then we write $D = 0$ and we say that D is *effective*. Irreducible submanifolds C_i of codimension 1 taken with the coefficient 1 are called *simple* Divizors. If in (4.18) all the $l_i \neq 0$, then the varyety $C_1 \cup \cdots \cup C_r$ is called the *support* of D and is denoted by $SuppD$. If in (4.18) the simple Divizors C_i are different we denote l_i by $v_{C_i}(D)$.

Now we define an operation of addition of Divizors. We observe that if we allow the coef?cients in (4.18) to take the value 0, any two Divizors can be written in the forma

$$D' = l'_1 C_1 + \cdots + l'_r C_r,\ D'' = l''_1 C_1 + \cdots + l''_r C_r$$

with the same $C_1, \ldots, C_r$. Then, by definition,

$$D' + D'' = (l'_1 + l''_1) C_1 + \cdots + (l'_r + l''_r) C_r$$

Thus, divisors on X forma a group isomorphic to a free module over $\mathbb{Z}$, whose generators are irreducible submanifolds of codimension 1 in X. This group is denoted by $DiV(X)$.

We now turn to the comparison of the function $f \in k[X]$, $f \neq 0$ of the divisor. Let C be a simple Divizor: We first compare each $f \in k(X)$, $f \neq 0$, an integer $v_C(f)$. If $X = \mathbb{A}^1$, then this will be an analogue of order zero (or pole) of the function f at a point. This can be done only with one restriction on the varyety X. Namely, we assume that X is smooth in codimension 1, that is, that the set of singular points of X has codimension ≥ 2. Let $C \subset X$ be an irreducible subvaryety of codimension 1 and U some affine open set consisting of simple points, intersecting with C and is such that C is determined in U by a local equation. Such a set U exists exists because of the constraint imposed on X. Thus, $a_C = (\pi)$ in $k[U]$. We prove that for any function $f \in k[X]$, $f \neq 0$, there is such a number $l \geq 0$, such that $f \in (\pi^l)$, $f \notin (\pi^{l+1})$. If this were not, if $f \in (\pi^l)$ for all $l > 0$, then $f \notin \cap \pi^l$, therefore $f = 0$.

The number l defined by us is denoted by $v_C(f)$. It has the properties

$$\left.\begin{aligned} v_C(f_1 f_2) &= v_C(f_1) + v_C(f_2) \\ v_C(f_1 + f_2) &\geq \min(v_C(f_1) : v_C(f_2)) \\ &for\ f_1 + f_2 \neq 0 \end{aligned}\right\} \tag{4.2}$$

which easily follow from the definition and irreducibility of C. If X is irreducible, then any function $f \in k(X)$ can be represented in the $f = g/h$, $g, h \in k[U]$. For $f \neq 0$ we set $v_C(f) = v_C(g) - v_C(h)$. From (4.19) it immediately follows that $v_C(f)$ does not depend on the representation of f in the forma f/g and that

4.19) is true for all $f \in k(X)$ and other zero. The definition of the number $v_C(f)$ given by us so far depended on the choice of the open set U, and therefore we should write $v_C^U(f)$ instead of $v_C(f)$. We will prove that in fact, $v_C^U(f)$ does not depend on U.

Suppose first that U is an affine open set, $V \subset U$, and $V \cap C \neq 0$. Then π is a local equation of C also in V, and it is obvious that, $v_C^V(f) = v_C^U(f)$. But if V is any open set satisfying the same conditions as U, then $U \cap C$ and $V \cap C$ are open in C and non-empty, and since C they have a non-empty intersection. Taking W to an affine neighborhood in $U \cap V$ of any point $\in U \cap V \cap C$, we obtain that, according to the previous remark $v_C^U(f) = v_C^W(f)$, $v_C^V(f) = v_C^W(f)$, then $v_C^U(f) = v_C^V(f)$. Thus, the correctness of the notation $v_C(f)$ is justi?ed. Note that if $X = \mathbb{A}^1$, if $C = x$ is a point with coordinate α, and $f \in k\left[\mathbb{A}^1\right] = k[T]$, then $v_x(f)$ coincides with the multiplicity of the root α of the polynomial $f(T)$, and the general de?nition essence copies this particular case.

If $v_C(f) = l > 0$, then say that function f has a zero of order l on submanifolds C. If $v_C(f) = -l < 0$, then f has a pole order l on submanifolds C. Observe that these concepts are de?ned for submanifolds of codimension 1 and not for points. For example, for the function x/y on $\mathbb{A}^2$ the point $(0,0)$ belongs to both the subvaryety of zeros $(x = 0)$ and the subvaryety of poles $(y = 0)$ of the function.

We now prove that a given function $f \in k(X)$ corresponds to only a finite number of irreducible submanifolds of codimensio 1 such that $v_C(f) \neq 0$. We first consider the case when X is an af?ne varyety and $f \in k[X]$. Then it follows from the definition that if C is not a component of the subvaryety $V(f)$, then $v_C(f) = 0$. If X is affine, as before, but $f \in k(X)$, then $f = ??/h$, g, $h \in k[X]$, and we saw that $v_C(f) = 0$ if C is not a component of $V(g)$ or $V(h)$. Finally, in general case, let $X = \cup U_i$ was a ?nite covering of X by af?ne open sets. Then any C intersects at least one U_i, therefore $v_C(f) \neq 0$ only for those C which are closures of such irreducible submanifolds $\tilde{C} \subset U_i$, such that $v_{\tilde{C}}(f) \neq 0$ in U_i. Since the numbers U_i and of $\tilde{C}$ in any U_i are finite and the number of C with $v_C(f) \neq 0$. So we can consider the Divizor

$$\sum v_C(f)\, C \tag{4.3}$$

where the sum is extended to all irreducible submanifolds of codimension 1 for which $v_C(f) \neq 0$. This is called the *Divizor of the f* and is denoted by (f).

A Divizor view $D = (f)$, $f \in k(X)$, is called main. If $(f) = \sum l_i C_i$ then to the Divizors $(f)_0 = \sum_{i: l_i > 0} l_i C_i$ and $(f)_\infty = -\sum_{i: l_i > 0} l_i C_i$ are *called Divizors of the zeros* and of *the poles* of f. Obviously, $(f)_0 \geq 0$, $(f)_\infty \geq 0$, $(f) = (f)_0 - (f)_\infty$.

We draw attention to some simple properties:

$(f_1 f_2) = (f_1) + (f_2)$, $(f) = 0$ if $f \in k$; $(f) \geq 0$ if $f \in k[X]$.

Let us prove that for a smooth irreducible varyety X the opposite is also true: if $(f) \geq 0$, then the function f is a regular on the varyety$X$. Let $x \in X$ a point at which f is non-regular. Then $f = g/h$, $h, g \in \mathcal{O}_x$ while $f \notin \mathcal{O}_x$. From the uniqueness of the decomposition into simple factors in $\mathcal{O}_x$ it follows that h and g can be chosen mutually simple in $\mathcal{O}_x$. Let π is a simple element of the ring $\mathcal{O}_x$which is included in h but not in g. In some affine neighbourhood U of a point x the varyety $V(\pi)$ is irreducible and has codimension 1. We denote its closure in X by C. Then obviously $v_C(f) < 0$. This result is also true if the varie X is a normal varyety, but we will not prove it here.

Since on a projektive irreducible varyety X a function that is regular at all points is a constant, then it follows from the just-proved result that if $f \geq 0$, then $f = \alpha \in k$ on a smooth projjective varyety X. In particular, on a smooth projektive irreducible varyety a rational function is uniquely determined by its Divizor to a constant factor: if $(f) = (g)$, then $\left(fg^{-1}\right) = 0$ and $f = \alpha g, \alpha \in k$

The main Divizors forma a subgroup $P(X)$ of the group $Div(X)$ of all Divizors. The factor group $Div(X)/P(X)$ called the *Divizor class group* and is denoted by $Cl(X)$. Divisors belonging to the same class adjacencies in $Div(X)/P(X)$ are called *equivalent:* $D_1 \sim D_2$ if $D_1 - D_2 = (f)$, $f \in k(X)$. The adjacency class in $Div(X)/P(X)$ are called *Divizor classes*. In the examples above we have:

$$1. Cl\left(\mathbb{A}^n\right) = 0 2. Cl\left(\mathbb{P}^n\right) = \mathbb{Z} 3. Cl\left(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_l}\right) = \mathbb{Z}^l$$

2. Locally Main Divizors. Suppose the varyety X is mooth. In this case, for any simple Divizor $C \subset X$ and any point $x \in X$ there is an open set U with $x \in U$ in which C is given by the local equation π. If D is any Divizor, $D = \sum l_i C_i$, and in U any of C_i is given by the local equation π_i, then we have $D = (f)$, $f = \prod \pi_i^{l_i}$. Thus, any point x has a neighbourhood in which D is main. From all such neighbourhoods we can choose a finite covering $X = \cup U_i$, and in any U_i we have $D = (f_i)$.

Obviously, the function f_i cannot be chosen arbitrarily: f_i are not identically zero.

and in $U_i \cap U_j$ the Divizors (f_i) and (f_j) coincide. As we saw above, it follows that the function $f_i f_j^{-1}$ is regular in $U_i \cap U_j$ and does not vanish there. If the system of functions $\{f_i\}$ corresponding to the sets of the covering $\{U_i\}$ satisfies the conditions $f_i f_j^{-1}$ is regular and does not vanish in $U_i \cap U_j$, then we will call consistent.

On the contrary, any coordinated system of functions defines a divisor on X. Indeed, for any simple divisor C we set $l_C = v_C(f_i)$ if $U_i \cap C \neq 0$, where f_i and C are treated as a function and a simple divisor in the varyety U_i. From the consistency of the system of functions it follows that this number does not depend on the choice of U_i. Obviously there is only finite number of such C such that $l_C \neq 0$, are the closures of irreducible divisor component f_i. Therefore we can consider the Divizor $D = \sum l_C C$. Obviously , it corresponds to a given system of functions $\{f_i\}$.

Finally, it is easy to find out when the system of functions $\{f_i\}$ and $\{g_j\}$ the corresponding to cover $\{U_i\}$ defines the same divisor as the system $\{V_j\}$, respectively, give one and the same Divizor. For this it is necessary and sufficient that in $U_i \cap V_j$ the functions $f_i g_j^{-1}$ were anywhere regular and do not vanish. A simple check is provided to the reader.

The assignment of divisors by systems of functions allows one to study their behavior with regular mappings. Let $\varphi : X \to Y$ regular mapping of smooth irreducible manifolds and D be a Divizor on Y. Suppose that $\varphi(X) Supp D$. We prove that with this restriction one can define the pre-image $\varphi^*(D)$ of the divisor D in the same way as the pre-image of a regular function is defined. First of all find out when it is possible to construct the pre-image of a rational function f on Y can be constructed, and and when it is not identically zero to X. For this it is enough that there is at least one point $y \in \varphi(X)$ in which f is regular and $f(y) \neq 0$. Then such points forma not an empty open set V. Now f is regular on V, and therefore $\varphi^*(f)$ regular function on $\varphi^{-1}(V)$ that does not vanish identically (in fact, nowhere). Since $\varphi^{-1}(V)$ is open in X, we see that $\varphi^*(f)$ determines a rational function in X. In terms of Divizors our condition on the mapping φ and the function f is reduced to the fact that $\varphi(X) Supp(f)$.

Now suppose that the divisor D is defined by a consistent system of the function $\{f_i\}$ and a coating $\{U_i\}$. We consider those U_i for which $\varphi(X) \cap U_i$ not empty, and we prove that $\varphi(X) \cap U_i Supp(f_i)$. For it follows from the irreducibility of X that

$\overline{\varphi(X)}$ is irreducible in Y. If $\overline{\varphi(X)} \cap U_i \subset Supp(f_i)$, then from the irreducibility of $\overline{\varphi(X)}$ and the fact that $\varphi(X) \cap U_i$ not empty, it follows that $\overline{\varphi(X)} \subset Supp(f_i)$. Finally, from the fact that $Supp(f_i) \cap U_i = SuppD \cap U_i$, of the irreducibility of $\varphi(X)$ and the fact that it intersects with U_i it follows that $\varphi(X) \subset SuppD$, contrary to what was suggested.

Therefore, for all U_i which intersect with $\varphi(X)$ the rational functions $\varphi^*(f_i)$ are de?ned in $\varphi^{-1}(U_i)$. The sets $\varphi^{-1}(U_i) = V_i$ for which $\varphi(X)$ intersects with U_i are open and forma a covering of X, and the functions $\varphi^*(f_i)$ forma a coordinated system, of functions that defines some divisor on X. Obviously this Divizor will not change if D another system of functions. The results Divizor is called the prototype Divizor D and is denoted by $\varphi^*(D)$.

In particular, if $\varphi(X)$ is dense in Y, then the pre-image of some Divizor $D \in Div(Y)$ is defined.

If D and D' are two Divizors on Y given on systems of functions of $\{f_i\}$ and $\{g_j\}$, corresponding to coverings $\{U_i\}$ and $\{V_j\}$, then the Divizor $D + D'$ is given by the system of functions $\{f_i g_j\}$ and the covering $\{U_i \cap V_j\}$. From this it follows at once that $\varphi^*(D + D') = \varphi^*(D) + \varphi^*(D')$, so that if $\varphi(X)$ is dense in Y, then φ^* defines a homomorphism

$$\varphi^* : DivY \to DivX$$

The main Divizor (f) is given by the system of functions $f_i = f$, and therefore $\varphi^*((f)) = (\varphi^*(f))$. Therefore φ^* maps $P(Y)$ into $P(X)$ and defines the homorphism $\varphi^* : Cl(Y) \to Cl(X)$.

As an application of the assignment of divisors by agreed systems functions we prove how it is possible to compare a divisor, not a function, but a forma from coordinates on a smooth projektive varyety. Let $X \subset \mathbb{P}^N$ and F be a forma in the coordinates in $\mathbb{P}^N$ that is not vanish identically on X. For any point $x \in X$ consider a forma G of the same degree as F, that $G(x) \neq 0$. Such formas exist: for example, if $x = (\alpha_0 : \cdots : \alpha_N$ and $\alpha_i \neq 0$, we can take $G = T_i^{degF}$. Then $f = F/G$ is a rational function on X and regular in an open set, in which $G \neq 0$. It is easy to see that there are such formas of G_i that open sets $U_i = X - X_{G_i}$ forma a covering of the varyety X. Just as easy believe that the functions of $f_i = F/G_i$ and the open subsets U_i forma a consistent system of functions and, therefore, define a divisor on X. Another choice of formas G_i will not change this divisor, which therefore depends depends only the forma F. It is called the

Divizor of F and is denoted by (F). Since the f_i are regular in the sets U_i, we have $F \geq 0$. If F_1 is another forma, $degF_1 = degF$, then $(F)-(F_1)$ is the Divizor of the rational function F/F_1. Therefore $(F) \sim (F_1)$ if $degF_1 = degF$.
In particular, all Divizors (L), where L is a linear formaa, are equivalent to each other. Evidently $Supp(L) = X_L$, the section of X by the hyperplane $L = 0$. Therefore they are called *hyperplane divizors section*.
Taking higher for F_1 the forma L^{degF} we get that $(F) \sim degF(L)$, where (L) hyperplane section divisor.
All arguments related to the definition of a divisor by a consistent system of functions can be generalized to arbitrary, not necessarily smooth manifolds. However, the possibility of setting a consistent system of functions must be taken for the definition of a divisor. The object to which we thus arrive is called a *locally main Divizor*.
More precisely, the locally main divisor on irreducible manifolds is a system of rational functions $\{f_i\}$ corresponding to open sets of the cover $\{U_i\}$ and satisfying the conditions:
1) the f_i are not equal identically 0 and
2) $f_i f_j^{-1}$ and $f_j f_i^{-1}$ are regular on $U_i \cap U_j$.
Moreover, the functions $\{f_i\}$ and $\{g_j\}$ and coverings $\{U_i\}$ and $\{V_j\}$, respectively, determine the same Divizor if $f_i g_j^{-1}$ and $f_i^{-1} g_j$ are regular in $U_i \cap V_j$.
Some function $f \in k(X)$ defines a locally main divisor if put $f_i = f$. Such Divizors are called main.
The product of locally main divisors defined by functions $\{f_i\}$ and $\{g_j\}$ by coverings $\{U_i\}$ and $\{V_j\}$, is called a divisor, given by the functions $\{f_i g_j\}$ and the cover $U_i \cap V_j$. All locally main Divizors forma a group, and the main Divizors a subgroup of it. The factor group is called the *Picard group* of the varyety X and is denoted by $Pic(X)$.
Any locally main Divizor has a carrier – it is the closed subvaryety consisting in the set in U_i of points at which f_i is not regular or to zero. Just as for Divizors on smooth manifolds, one can de?ne the preimage of a locally main Divizor D on Y under the regular mapping $\varphi : X \to Y$ if $\varphi(X)$ is notcontained in $SuppD$.
We note one important special case. If X is a smooth varyety and Y is not necessarily smooth manifold, and then any such Divizor D on X for which $SuppDY$ determines a locally main Divizor $\tilde{D}$ on Y. To see this we have to consider the

embedding $\varphi : Y \to X$ and to set $\tilde{D} = \varphi^*(D)$. We will call $\tilde{D}$ the restriction of D to Y and denote it by $\rho_Y(D)$. From the de?nition it follows that for main Divizors $\rho_Y((f)) = (f)$. where f the restriction of the function f on Y.

The difference between divisors and locally main divisors manifests itself only in the case of nonsmooth manifolds.

3. How to Shift the Support of a Divizor Away from Points

Theorem 1. For any Divizor D on a smooth varyety X and finitely many points $x_1, \dots, x_m \in X$ there exists a Divizor D' such that $D' \sim D$, $x_i \notin SuppD'$ $(i = 1, \dots, m)$.

We can assume that D is a simple Divizor, otherwise it would be sufficient apply the theorem to each of its components. Choose an affine open set in X containing points $x_1, \dots, x_m$. It suffices to prove the theorem for this set. so we can assume that X is an affine varyety. Applying induction on the number m, we can assume that $x_1, \dots, x_m \notin SuppD$. $x_{i+1} \notin SuppD$. It remains to construct such a divisor D' that $D' \sim D$, $x_1, \dots, x_{i-1} \notin SuppD'$. Consider some local equation equation π' simple Divizor D in a neighbourhood of x_{i-1}. We prove that π' one can chosen so that $\pi' \in k[X]$ (by assumption, X is affine). Really, π' regular at the point x_{i+1}. and therefore, if $(\pi')_\infty = \sum k_l F_l$. then $x_{i+1} \notin F_l$. Therefore for each l there is a function $f_l \in k[X]$ equal to zero on F_l and that $f_l(x_{i+1}) \neq 0$. Function it $\pi = \pi' \prod f_l^{k_l}$ is regular on X and is a local equation of D in a neighbourhood of x_{i+1}. Since the condition $x_j \notin SuppD \cup x_1 \cup \dots \cup x_{j-1} \cup x_{j+1} \cup \dots \cup x_i$ $(j = 1, \dots, i)$, for any $j = 1, \dots, i$ there exists a function $g_j \in k[X]$ that $g_j|_D = 0$, $g_j(x_l) = 0$ $(l = 1, \dots, j-1, j+1, \dots, i)$, $g_j(x_j) \neq 0$.

We consider the function

$$f = \pi + \sum_{j=1}^{i} \alpha_j g_j^2, \alpha_j \in k.$$

and choose the constants α_j so that

$$f(x_j) \neq 0 (j = 1, \dots, i) \tag{4.4}$$

It's enough to take $\alpha_j \neq -\pi(x_j)/g_j(x_j)^2$.Such that all $g_{j|D} = 0$. in the local rings $\mathcal{O}_{x_{i+1}}$ we have $g_j \equiv 0(\pi)$ and $\sum \alpha_j g_j^2 = \pi^2 h$, $h \in \mathcal{O}_{x_{i+1}}$; $f = \pi(1 + \pi h)$.

Since $(1 + \pi h)(x_{i+1}) = 1$, it follows that f is a local equation of D in a neighbourhood of x_{i+1}. Therefore $(f) = D + \sum r_s D_s$, and no one of the simple Divizors of

D_s passes through x_{i+1}. This means that if we put $D' = D - (f)$, then $x_{i+1} \notin SuppD'$. Next, (4.18) proves that $x_j \notin Supp(f)$ $(j = 1, \ldots, i)$, therefore the Divizor D' satis?es the conditions of the theorem.

Here is a first application of Theorem 1 we have de?ned the inverse image $f^*(D)$ of a Divizor D of a varyety X under a regular mapping $f : Y \to X$, provided that $f(Y)SuppD$.

Theorem 1 allows us to replace the divisor D with an equivalent Divizor D' such that $x \notin SuppD'$, where x is an arbitrarily chosen point of $f(Y)$. Then automatically $f(Y)SuppD'$ and the preimage $f^*(D')$ is de?ned. This proves that we can, without any restrictions on the regular mapping f determine the preimage of the class of Divizor $C \in Cl(X)$. To do this, choose in C such a Divizor D that $f(Y)SuppD$ and consider a class on Y containing the Divizor $f^*(D)$. It is easy to verify that in this way we get a homomorphism

$$f^* : Cl(X) \to Cl(Y)$$

In other words, $Cl(X)$ is a functor from the category of irreducible smooth algebraic manifolds to the category of Abelian groups.

4. Divizors and Rational Mappings. Comparison of the functions of divisors is useful for the study of rational mappings of manifolds into projektive space. Let X be a smooth manifold and $\varphi : X \to \mathbb{P}^n$ it's rational mapping. Find out at which points the maps φ is not regular.

Rational mapping is given by the formaulas

$$\varphi = (f_0 : \cdots : f_n), f_i \in k(X) \tag{4.5}$$

and we can assume that none of the functions f_i is identically zero on X. Let

$$(f_i) = \sum_{j=1}^{m} k_{ij} C_j$$

where the C_j- are simple Divizors. At the same time, we assume that some k_{ij} is zero.

To find out if φ is regular at a point $x \in X$ we define C_j by the local equation π_j at x. Then

$$(f_i) = \left(\prod_j \pi_j^{k_{ij}} \right) u_i, u_i \in \mathcal{O}_x, u_i(x) \neq 0.$$

Due to the uniqueness of the decomposition into simple factors in $\mathcal{O}_x$ there is

a common greatest divisor of d elements $f_0, \dots, f_n$ that is, an element $d \in k(X)$ such that $f_i d^{-1} \in \mathcal{O}_x$ and if $d_i \in k(X)$ is such $f_i d_1^{-1} \in \mathcal{O}_x$then $d_1 | d$, that is $dd_1^{-1} \in \mathcal{O}_x$.
Since the local equations of irreducible manifolds are simple elements in $\mathcal{O}_x$, then

$$d = \prod \pi_j^{k_j}, \ k_j = \min_{i=0,\dots,n} k_{ij}.$$

The mapping φ is regular at the point x if there exists such a function $g \in k(X)$, that $f_i g^{-1} \in \mathcal{O}_x$ $(i = 0, \dots, n)$ and the $\left(f_i g^{-1}\right)(x)$ are not all zero. In view of the definition of the common greatest divisor, this implies that $g|d$. If $d = g \circ h$, $h \in \mathcal{O}_x$ and $h(x) = 0$, then $h| \left(f_i g^{-1}\right)$, because all the $\left(f_i g^{-1}\right)(x) = 0$. Thus, only a function g such that $d = g \circ h$, $h(x) \neq 0$, can satisfy the necessary conditions. Then $f_i g^{-1} = \left(f_i g^{-1}\right)$, that is,

$$f_i g^{-1} = \left(\prod_j \pi_j^{k_{ij} \ k_j}\right)(u_i h).$$

and map φ regularly if and only if not all functions $\prod_j \pi_j^{k_{ij}-k_j}$ vanish at x. To translate this answer into the language of divisors, we call the common greatest divisor of divisors $D_i = \sum k_{ij} C_j$ $(i = 1, \dots n)$ Divizor

$$\text{g.c.d. } (D_1, \dots, D_n) = \sum k_j C_j, \ k_j = \min_{i=1,\dots,n} k_{ij}.$$

Obviosly, that $D_i' = D_i - g.c.d.(D_1, \dots, D_n) \geq 0$, and the Divizors D_i' have not common components. Suppose in particular, let us put

$$D = g.c.d\left((f_0), \dots, (f_n)\right), D_i' = (f_i) - D.$$

Then in some neighbourhood of x

$$\left(\prod_j \pi_j^{k_{ij}-k_j}\right) = D_i'$$

and we can say that the mapping φ is regular at x then and only when not all the $SuppD_i'$ manifolds pass through this point. So Shafareevich have proved the following result.

Theorem 2. The rational mapping (4.18) is non-regular precisely at the points of the set

$$\cap SuppD_i', \ D_i' = (f_i) - g.c.d\left((f_0), \dots, (f_n)\right) (i = 0, \dots, n).$$

Since the Divizors D'_i do not have common irreducible components, the set $\cap SuppD'_i$. is of codimension ≥ 2.

Remark. The Divizors D'_i can be interpreted as the inverse images of the hyperplanes $x_i = 0$ under the mapping $\varphi : X \to \mathbb{P}^n$. For if $x \notin \cap SuppD'_i$ and in $D = (h)$ a neighbourhood of x, then in the same neighbourhood a regular mapping is given by the formaulae:

$$\varphi = (f_0/h : \cdots : f_n/h).$$

The inverse image of the hyperplane $x_i = 0$ has the local equationf_i/h, hence coincides with D'_i.

More generally, if $\lambda = (\lambda_0 : \cdots : \lambda_n)$ and $E_\lambda \subset \mathbb{P}^n$ is the hyperplane $\sum \lambda_i x_i = 0$, then

$$\varphi^* (E_\lambda) = \left(\sum \lambda_i f_i\right) - D.$$

5. The Space Associated with a Divizor. The fact that all polynomials $f(t)$ are degree $\leq n$ forma a vector space of finite dimension, can be interpreted as follows in terms of divisors. Denote by x_∞ an infinitely distant point on the projektive line $\mathbb{P}^1$ with coordinate t. The polynomial in t of degree l has a pole of order l at x_∞ and has no other poles. Therefore the condition $degf \leq n$ can be expressed in the following: the Divizor $(f) + nx_\infty$ is effective.

Similarly, for an arbitrary Divizor D on a smooth varyety X we can consider a set consisting of zero and those funktions $f \in k(X)$, $f \neq 0$, for which

$$(f) + D \geq 0. \qquad (1.6)$$

This set is a linear space over the field k with respect to ordinary operations on functions. Indeed, if $D = \sum n_i C_i$, then (4.18) is equivalent to

$v_{C_i}(f) \geq -n_i$, $v_C(f) \geq 0$ for $C \neq C_i$

in view of which our statement immediately follows from the formaulas.

The space of functions satisfying conditions (4.18) is called the space associated with the Divizor D and is denoted by $\mathcal{L}(D)$.

The analogue of the fact that polynomials of degree $\leq n$ forma a finite-dimensional space is that the space $\mathcal{L}(D)$ is finite-dimensional, if D is any divisor, and X is a projektive varyety.

The dimension of the space $\mathcal{L}(D)$ is also called the dimension of the Divizor D and is denoted by $l(D)$.

Theorem 3. *Equivalent Divizors have equal dimensions.*

Let $D_1 \sim D_2$: this means $D_1 - D_2 = (g)$. $g \in k(X)$. If $f \in \mathcal{L}(D_1)$. then $f + D_1 \geq 0$. From this it follows that $(fg) + D_2 = f + D_1 \geq 0$. that is. $fg \in \mathcal{L}(D_2)$. $g\mathcal{L}(D_1) = \mathcal{L}(D_2)$. Thus. multiplication of all functions $f \in \mathcal{L}(D_1)$ the function g is determined by an isomorphism of the spaces $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$, and the theorem follows.

We see that it is possible. therefore. to speak of the dimension $l(C)$ of the class of Divizor C. meaning by this the common dimension of all the divisors of this class. This number has the following meaning. If $D \in C$. $f \in \mathcal{L}(D)$. then the Divizor $D_f = (f) + D$ is effective. Obviously. $D_f \sim D$. therefore $D_f \in C$. On the contrary, any effective divisor $D' \in C$ has the forma D_f, where $f \in \mathcal{L}(D)$. Obviously. if X is projektive. then the function f is determined by the Divizor D_f uniquely to a constant factor. Thus, we can establish a one-to-one correspondence between effective divisors of class C and points of the $(l(C) - 1)$-dimensional projektive space $\mathbb{P}(\mathcal{L}(D))$ corresponding to divizor D. (recall that the projektive space $\mathbb{P}(L)$ corresponding to vector space L consists of all direct speces L).

The space $\mathcal{L}(D)$ when defining rational mappings by Divizors. If

$$\varphi = (f_0 : \cdots : f_n) : X \to \mathbb{P}^n \tag{4.7}$$

is a rational mapping and,

$$D = g.c.d.((f_0), \ldots (f_n))D_i = (f)_i - D \tag{4.8}$$

then $D_i \geq 0$. hence all the $f_i \in \mathcal{L}(-D)$.

The choice of the functions φ depends on the chosen system of projektive coordinates in $\mathbb{P}^n$. Therefore. in an invariant way. the map $\varphi(X)$ there corresponds. to the set of all functions $\sum_{i=0}^{n} \lambda_i f_i$ that are linear combinations of the f_i. These functions forma a linear subspace $M \subset \mathcal{L}(-D)$. Further we will assume that $\varphi(X)$ is not contained in any proper linear subspace in $\mathbb{P}^n$. Then $\sum \lambda_i f_i \neq 0$ on X if doesn't all the $\lambda_i = 0$. The set of effective divisors corresponding to such a set of functions, i.e.. the Divizors $M(g) + D$. $g \in M$. is called a *linear system* of Divizors. If $M = \mathcal{L}(-D)$, then the linear system is called complete. The sense of Divizors $(f) - D$, $f \in M$, is very simple: these are types of the divisor of hyperplanes in $\mathbb{P}^n$ under the map φ. Thus, we can construct all rational mappings of a given smooth varyety X into different projektive spaces. For this we need to take an arbitrary Divizor D. and in the space $\mathcal{L}(-D)$ a linear ?nite-dimensional subspace M. If $f_0, \dots, f_n$ is its basis. then formaulas (4.19) will give desired mapping. Notice that Divizors D_i for thise $f_i \in \mathcal{L}(-D)$ have an additional property: they have no common components.

Since the multiplication of all functions f_i by the common factor $g \in k(X)$ does not change the mapping φ, and a Divizor D is changed to the equivalent Divizor $(g) + D$. the rational mapping invariantis the class of the Divizor D of the rational mapping. Thus. we have the following a way to constructing all such rational mappings φ of a varyety X into the projektive space $\mathbb{P}^m$ that $\varphi(X)$ is not contained in any proper subspace in $\mathbb{P}^m$: choose an arbitrary class of Divizors on X. and for any Divizor D of this class in the space $\mathcal{L}(-D)$ there is a linear ?nite-dimensional subspace M such that the effective Divizors $(f) - D$ have no common components.

If $f_0, \dots, f_n$ is a basis of M. then our mapping is given by formaula (4.19). Of course. it may turn out that $\mathcal{L}(-D) = 0$ or that all the Divizors $(f) - D$, $f \in \mathcal{L}(-D)$ have a common component, then this class of Divizor does not lead to any mapping.

We draw attention to one interesting property of the resulting picture. Among all rational mappings corresponding to a given class C, there is one a maximal: that which is obtained if taking for M the whole space $\mathcal{L}(-D)$. $D \in C$. (Here rely on the unproved theorem that the space $\mathcal{L}(-D)$ is ?nite-dimensional.)

All other mappings corresponding to this class are obtained by constructing the composing this mapping with various projection mappings. Indeed. if $\varphi = (f_0 : \dots : f_N)$ and say. $\psi = (f_0 : \dots : f_n)$, $n < N$, then $\psi = \pi\varphi$. where $\pi\,(x_0 : \dots : x_N)$ $(x_0 : \dots : x_n)$ is a design that we now consider as a rational mapping.

Let us see how this scheme works if take the projektive space $\mathbb{P}^m$ for X. We know that $Cl(\mathbb{P}^m) \simeq \mathbb{Z}$ and this class C_l, corresponding to an integer l consists of the Divizors of degree l.

Obviously, if $l > 0$, $D \in C_l$, then $\mathcal{L}(-D) = 0$. If $l \leq 0$, then we can take for $- - D$ the Divizor $- - lE$. where E is the Divizor of an infinity distant hyperplane $x_0 = 0$. In this case $\mathcal{L}(-lE)$ consists of polynomials of degree $\leq -l$ in inhomogeneous coordinates $x_1/x_0, \dots, x_m/x_0$. If multiply formaulas of the resulting for the mapping by x_0^l, we obtain the Veronese mapping $v_l : \mathbb{P}^m \to \mathbb{P}^{\nu_{l,m}}$.

Thus, we see that any rational mapping of the space $\mathbb{P}^m$ is obtained by composing the Veronese map and the design.

6. Divizors on Curves

1. Consider a projektive smooth curve. A Divizor on X is, therefore, a linear combination of points $D = \sum k_i x_i$, $k_i \in \mathbb{Z}$, $x_i \in X$. The degree of Divizor D is the number $degD = \sum k_i$.

Theorem 1. If $f : X \to Y$ is a regular mapping of smooth projektive curves and $f(X) = Y$, then $degf = degf^*(y)$ for any point $y \in Y$.

In Theorem 1 $f^*(y)$ is Divizor on X that is the inverse image of the Divizor on Y consisting of the point y with the coef?cient 1. Then, $degf$ is the number of preimages of any point $y \in Y$ (taken with right multiplicities). This makes the intuitive sense of the degree of mapping f clear. It proves how many times X covers Y when the map is f.

Corollary. The degree of a main Divizor on a smooth projektive curve X is equal to zero.

Indeed,any non-constant function $f \in k(X)$ defines a regular map $f : X \to \mathbb{P}^1$. At the same time for the point $f^*(0) = (f)_0$ for $0 \in \mathbb{P}^1$ – this immediately follows from the definition of both Divizors. Similarly $f^*(\infty) = (f)_\infty$. By Theorem 1.

$$deg(f) = deg(f)_0 - deg(f)_\infty = degf^*(0) - degf^*(\infty) = degf - degf = 0$$

If X and Y are two manifolds of the same dimension and if f is a regular mapping $f : X \to Y$ such that $f(X)$ is dense in Y, then it determines an embedding $f^* : k(Y) \to k(X)$ utilizing this we shall henceforth regard $k(Y)$ as a sub?eld of $k(X)$ (that is, for $u \subset k(Y)$ we write u instead of $f^*(u)$ when this cannot lead to misunderstandings).

Theorem 1 follows from two results. To state them we introduce the following notation. Let $x_1, \ldots, x_r$ is points on the curve . We set

$$\widetilde{\mathcal{O}} = \bigcap_{i=1,\ldots,r} \mathcal{O}_{x_i} \tag{4.9}$$

Thus, $\widetilde{\mathcal{O}}$ consists of the functions that are regular at all the points $x_1, \ldots, x_r$. If $\{x_1, \ldots, x_r\} = f^{-1}(y)$, $y \in Y$, then the ring $\mathcal{O}_y$, which we have agreed above to regard as a subring of $k(X)$, is contained in $\widetilde{\mathcal{O}}$.

Theorem 2. $\widetilde{\mathcal{O}}$ is a prin?cipal ideal ring with finitely many simple ideals.

There exist elements $t_i \in \widetilde{\mathcal{O}}$ such that

$$v_{x_i}(t_j) = \delta_{ij}, 1 \le i, j \le r. \tag{4.10}$$

If $u \in \tilde{\mathcal{O}}$, then
$$u = t_1^{l_1} \cdots t_r^{l_r} v, \tag{4.11}$$

where $l_i = v_{x_i}(u)$and v is invertible in $\tilde{\mathcal{O}}$.

Theorem 3. If $\{x_1, \ldots, x_r\} = f^{-1}(y)$, then $\tilde{\mathcal{O}}$ is a free module over $\mathcal{O}_y$ and $\tilde{\mathcal{O}} \simeq \mathcal{O}_y^n$, where $n = deg f$.

Let us ?rst prove how Theorem 1 follows from Theorems 2 and 3. Let t be a local parameter at y, and $\{x_1, \ldots, x_r\} = f^{-1}(y)$.

According to Theorem 2, $t = t_1^{l_1} \cdots t_r^{l_r} v$, where $l_i = v_{x_i}(t)$. Recalling the de?nition of the inverse image of a Divizor we see that

$f^*(y) = \sum l_i x_i$ and $deg f^*(y) = \sum_{i=1}^r l_i$.

Since the elements $l_1, \ldots, l_r$ are pairwise cosimple in $\tilde{\mathcal{O}}$, we have

$$\tilde{\mathcal{O}}/(t) \simeq \bigoplus_{i=1}^{r} \tilde{\mathcal{O}}/\left(t_i^{l_i}\right)$$

It is easy to see that any element we $w \in \tilde{\mathcal{O}}$ has a unique representation in the forma

$$w \equiv \alpha_0 + \alpha_1 t_i + \cdots + \alpha_{l_i - 1} t_i^{l_i - 1} \left(mod\ t_i^{l_i}\right), \alpha_i \in k. \tag{4.12}$$

For if we have already got the representation
$$w \equiv \alpha_0 + \alpha_1 t_i + \cdots + \alpha_{s-1} t_i^{s-1} (mod\ t_i^s)$$

Then
$$v = t_i^{-s}(w - \alpha_0 - \cdots - \alpha_{s-1} t_i^{s-1}) \in \tilde{\mathcal{O}} \subset \mathcal{O}_{x_i}.$$

We set $v(x_i) = \alpha_s$. Then $\nu_{x_i}(v - \alpha_s) > 0$, and from Theorem 2 it follows that $v \equiv \alpha_s (mod\ t_i)$, that is,
$$w \equiv \alpha_0 + \alpha_1 t_i + \cdots + \alpha_{s-1} t_i^{s-1} + \alpha_s t_i^s \left(mod\ t_i^{s+1}\right).$$

This proves (4.12) by induction.

From the representation (4.12) it follows that $dim\ \tilde{\mathcal{O}}/\left(t_i^{l_i}\right) = l_i$Therefore

$$dim \tilde{\mathcal{O}}/(t) = \sum_{i=1}^{r} l_i. \tag{4.13}$$

When we now apply (1.13) it follows that $\tilde{\mathcal{O}}/(t) \simeq (\mathcal{O}_y/(t))^n$. But I is a local parameter at y, therefore

$$\mathcal{O}_y/(t) \simeq k.\,dim\tilde{\mathcal{O}}/(t) = n = deg f \qquad (1.14)$$

Now (??) and (4.14) prove Theorem 1.

Proof of Theorem 2 .We denote by u_i a local parameter at x_i. Then x_i occurs in the Divizor (u_i) with the coef?cient 1 that is, $(u_i) = x_i + D$, where x_i does not occur in D. I can shift the support of D away from $x_1, \ldots, x_r$, that is, we can ?nd a function f_i such that these points do not occur in $D + (f_i)$. This means that for $t_i = u_i f_i$ the relations (4.19) hold. Let $u \in \tilde{\mathcal{O}}$. We set $v_{x_i}(u) = l_i$. By hypothesis, $l_i \geq 0$. For the element $\nu = ut_1^{-l_1} \cdots t_r^{-l_r}$ we have $v_{x_i}(\nu) = 0$, for all $i = 1, \ldots, r$, from which it follows that $v \in \tilde{\mathcal{O}}$ and $v^{-1} \in \tilde{\mathcal{O}}$ So we obtain a representation (4.20) for u. It remains to verify that $\tilde{\mathcal{O}}$ is a plincipal ideal ring. Let a be an ideal of $\tilde{\mathcal{O}}$. We set $l_i = \inf_{u \in a} v_{x_i}(u)$ and $= t_1^{l_1} \cdots t_r^{l_r}$. Then $ua^{-1} \in \tilde{\mathcal{O}}$ that is, $a \subset (a)$.

Let us prove that $a = (a)$. To do this we denote by a' the set of functions ua^{-1}, $u \in a$. Evidently a' is an ideal of $\tilde{\mathcal{O}}$ and $\inf_{u \in a'} v_{x_i}(u) = 0$. Hence for any $i = 1, \ldots, r$ there exists a $u_i \in a'$ for which $v_{x_i}(u_i) = 0$, that is, $v_i(x_i) = 0$. An obvious veri?cation proves that $v_{x_i}(c) = 0$ $i = 1, \ldots, r$ for the element $c = \sum_{j=1}^{r} u_j t_1 \ldots \hat{t}_j \ldots t_r \in a'$ (the symbol $\hat{t}_j$ indicates that the corresponding fajctlır is absent). This means that $c^{-1} \in \tilde{\mathcal{O}}$, therefore $a'^{-\tilde{\mathcal{O}}}$, $a = (a)$. This proves the theorem.

Now we turn to the proof of Theorem 3. First of all, we prove that $\tilde{\mathcal{O}}$ is a module of ?nite type over $\mathcal{O}_y$. For this purpose we recall that according to Theorem 11 the mapping f is ?nite. There fore the point y has an af?ne neighbourhood V such that the curve $U = f^{-1}(V)$ is also affine and that the ring $A = k[U]$ is a module of finite type over $B = k[V]$. As always, the embedding $B \subset A$ is effected by the mapping f^*.

Lemma. In the previous notation $\tilde{\mathcal{O}} = A\mathcal{O}_y$ even if Y is not a normal curve. For if $\varphi \in \tilde{\mathcal{O}}$ and z_i are the poles of φ on U, then $f(z_i) = y_i \neq y$. There exists a function $h \in B$ such that $h(y) \neq 0$, $h(y_i) = 0$, and $\varphi h \in \mathcal{O}_{z_i}$, hence $\varphi h \in A$ Since $h^{-1} \in \mathcal{O}_y$, we have $\varphi \in A\mathcal{O}_y$. So we have proven that $\tilde{\mathcal{O}} \subset A\mathcal{O}_y$. The reverse Einclusion IS obvious. and the lemma is proved.

Now we can complete the proof of Theorem 3. Clearly. generators of the module A over $k[Y]$ are at the same time generators of $A\mathcal{O}_y$ over $\mathcal{O}_y$. Therefore $\tilde{\mathcal{O}}$ is a module of ?nite type. By the main theorem on modules over a main ideal ring. $\tilde{\mathcal{O}}$ is a direct sum of a free module and a torsion module. However $\mathcal{O}_y$ and $\tilde{\mathcal{O}}$ are contained in the ?eld $k(X)$. from which it follows that this torsion module is zero and that $\tilde{\mathcal{O}} \simeq \mathcal{O}_y^m$ for some m.

It remains to determine m that is. the rank of $\tilde{\mathcal{O}}$. It IS equal to the maginum number of linearly independent elements over $\mathcal{O}_y$ contained in $\tilde{\mathcal{O}}$. Since linear independence over a ring and over its field of fractions is one and the same thing. and since the ?eld of fractions of $\mathcal{O}_y$. is $k(Y)$. we see that m is equal to the maximal number of linearly independent elements of $\tilde{\mathcal{O}}$ over $k(Y)$.

By hypothesis. $[k(X) : k(Y)] = n$, so that necessarily $m \le n$. It remains to prove that $\tilde{\mathcal{O}}$ contains n linearly independent elements relative to $k(Y)$ Let $\alpha_1, \ldots, \alpha_n$ be a basis of the extension $k(X)/k(Y)$. We denote by l the maximum order of the poles of the functions α_i, at the points x and by t a local parameter of y.Evidently the functiohs $\alpha_i t^l$ are regular at these points. hence are contained in $\tilde{\mathcal{O}}$. Consequently. they are linearly independent over $k[Y]$. This completes the proof of the theorem.

7. Bezout's Theorem on Curves. Here we give the simplest applications of the theorem on the degree of a main Divizor. They are very special cases of more general theorems. which we shall prove in connection with the theory of intersection indices. However. it is convenient to give an account of these simple cases now. because they will be useful for us in the next subsection.

Let X be a smooth projektive curve, $X \subset \mathbb{P}^n$, F a forma in the point coordinates of $\mathbb{P}^n$ that is not identically zero on X and x a point on X.

We have introduced the Divizor (F) of F on X. The degree $\deg(F)$ of this Divizor is also denoted by $(X.F)$ and is called the intersection index of X with the hypersurface $\mathbb{P}^n_F$.

Theorem 1 leads at once to an important consequence: this number is one and the same for all formas of the same degree.

For if $degF = degF_1$. then $f = F/F_1 \in k(X)$. From the de?nition of the Divizor (F) it follows at once that $(F) = (F_1) + (f)$. hence $(F)(F_1)$. By the corollary to Theorem 1. $degF = degF_1$.

To find out how the number (X, F) depends on the degree of the Forma F it is suf?cient to take for F any forma of degree $m = deg F$. In particular, we may set $F = L^m$, where L is a linear forma. Then

$$(X, F) = m\,(X, L) = (degF)(X, L) \tag{4.15}$$

Finally, we explain the meaning of the number (X, L). We have introduced the concept of the degree $degX$ of a curve X as then maximum number of points of intersection of X with a hyperplane not containing X.

Since $(X, L) = \sum_{L(x)} v_x(L)$, we have $\deg X \leq (X, L)$. Let us ?nd out when $v_x(F) = 1$ for the case of an arbitrary forma F. By virtue of the additivity of the function $v_x(F)$ it is sufficient to consider the case of an irreducible forma.

Lemma. Let $X \subset \mathbb{P}^n$, F an irreducible forma, and $Y = \mathbb{P}^n_F$§. The equality $v_x((L)) = 1$ is equivalent to the fact that $F(x) = 0$, and $\Theta_{x,Y} \cdot \Theta_{x,X}$ Both these spaces are regarded as subspaces of $\Theta_{x,\mathbb{P}^n}$.

The proof comes from a comparison of some de?nitions. Let G be a forma for which $G(x) \neq 0$, $degF = degG$. By de?nition, $v_x(F) = v_x(f)$, where $f = (F/G)|_X$. We know that $v_x(f) > 1$ is equivalent to the fact that $f \in m_x^2$, or, what is the same, $d_x f = 0$. But $d_x f \in \Theta^*_{x,X}$ is also the restriction to $\Theta_{x,X}$ of the differential $d_x(F/G)$ the function F/G, which is rational on $\mathbb{P}^n$ and regular at x. Thus, $v_x(F) > 1$ is equivalent to $d_x(F/G) = 0$ on $\Theta_{x,X}$. Furthermore, F/G is a local equation of Y in a neighbourhood of x in which $G \neq 0$. Therefore $d_x(F/G) = 0$ is the equation of OH, and $d_x(F/G) = 0$ on $\Theta_{x,X}$ if and only if $\Theta_{x,Y} \supset \Theta_{x,X}$. We apply this to calculate the intersection index (X, L).

Since the number (X, L) is one and the same for all linear formas L, the number of points $x \in X$ for which $L(x) = 0$ assumes its maximum when all the $v_x(L) = 1$. By the lemma this is equivalent to the fact that the hyperplane L does not touch X at any point. Taking for L such a linear forma we ?nd that

$$degX = (X, L) \tag{4.16}$$

It only remains to verify that linear formas with the required property actually exist. This is easily done by means of arguments we have used many times: in the product $X \times \tilde{\mathbb{P}}^n$ (where $\tilde{\mathbb{P}}^n$ is the space of hyperplanes in $\mathbb{P}^n$) we consider the set Γ of pairs (x, ξ) such that ξ touches X at x. A standard application of the

theorem on the dimension of ?bres of mappings then proves that the image of Γ under the projection $X \times \widetilde{\mathbb{P}}^n \to \widetilde{\mathbb{P}}^n$ is of codimension ≥ 1.
Comparing (4.18) and (4.19) we obtain the relation

$$(X.F) = degF \bullet degX \qquad (4.17)$$

which is called *Bezout's theorem.* This theorem has many applications in elementary geometry.

8. Cubic Curves. From the corollary to Theorem 1 it follows that all equivalent Divizors on a smooth projektive curve have the same degree. Hence we can speak of the degree of a Divizor class. We have therefore the homomorphism

$$deg : Cl(X) \to \mathbb{Z}$$

whose image is the whole group $\mathbb{Z}$ and whose kernel consists of the classes of degree zero and is denoted by $Cl^{\circ}(X)$. The role of this group will already be clear from the following result.

Theorem 4. A smooth projektive curve X is rational if and only if

$$Cl^{\circ}(X) = 0.$$

For if $X \approx \mathbb{P}^1$ for $n = 1$. There we have seen that $Cl^{\circ}(\mathbb{P}^1) = \mathbb{Z}$ and hence $Cl^{\circ}(\mathbb{P}^1) = 0$. Conversely, let $Cl^{\circ}(X) = 0$. This means that any Divizor of degree zero is main. In particular, if $x, y \in X$, $x \neq y$, then there exists a function $f \in k(X)$ such that $x - y = (f)$. Regarding f as a mapping $X \to \mathbb{P}^1$ we deduce from Theorem 1 that $k(X) = k(f)$, that is, f is a birational isomorphism. Since X and $\mathbb{P}^1$ are smooth projektive curves, f is an isomorphism.
Now we analyse the simplest case, when $Cl^{\circ}(X) \neq 0$.These are plane smooth projektive curves of degree 3. We have seen such curves need not be rational: for example, the curve with the equation $x^3 + y^3 = 1$ is nonrational. We prove that all plane smooth projektive curves of degree 3 are non-rational. We shall now make use of this fact.

Theorem 5. We choose an arbitrary point x_0 on a smooth projektive plane curve X of degree 3 and associate with any point $x \in X$ the class C_x containing the Divizor $x - x_0$. The mapping $x \to C_x$ determines a one-to-one correspondence between points $x \in X$ and classes $C \in Cl^{\circ}(X)$.
If $C_x = C_y$, $x - x_0 \sim y - x_0$and $x \sim y$. From the proof of Theorem 4 it follows

that for any $x \neq y$ this would lead to the curve X being rational, whereas we know that it is not.

It remains to prove that in any class C of degree zero there is a Divizor of the forma $x - x_0$. To begin with, let D be any effective Divizor. We prove that there exists a point $x \in X$ such that

$$D \sim x + lx_0 \tag{4.18}$$

If $degD = 1$, then (4.18) is true with $l = 0$. If $degD > 1$, then $D = D'^{+y}$, $degD'^{=degD-1}$, $D'^{>0}$. Applying induction we may assume that (4.18) is proved for $D' : D'^{\sim z+mx_0}$. Then $D \sim y + z + mx_0$. If we can find a point x such that

$$y + z \sim x + x_0 \tag{4.19}$$

then (4.18) follows.

First let $y \neq z$. We draw the line through these points with the equation $L = 0$. By Bezout's theorem $(L.X) = 3$, and hence

$$(L) = y + z + u,\ u \in X. \tag{4.20}$$

Next we suppose that $u \neq x_0$, and we draw the line through u and x_0, with the equation $L_1 = 0$. As in (4.20) we ?nd that $(L_1) = u + x_0 + x$. Since $(L) \sim (L_1)$ we have $y + z + u \sim u + x_0 + x$., hence (4.19) follows.

We still have to analyse the cases when $y = z$ or $u = x_0$ If $y = z$, then we draw the tangent to X at y. Let $L = 0$. be its equation. $v_y((L)) \geq 2$ and therefore $(L) = 2y + u$. Thus, (4.19) also holds in this case. The case $u = x_0$ is treated similarly.

Now let $degD = 0$. Then $D : D_1 - D_2$, $D_1 \geq 0$, $D_2 \geq 0$, $degD_1 = degD_2$. Applying (4.18) to D_1 and D_2 we see that $D_1 \sim y + lx_0$, $D_2 \sim z + lx_0$with one and the same l, because $degD_1 = degD_2$. Therefore

$$D = D_1 - D_2 \sim y - z,$$

and it is suf?cient to ?nd a point x for which $y - z \sim x - x_0$. This is equivalent to $y + x_0 \sim z + x$ and is the same as (4.19) apart from the notation.

4. The Dimension of a Divizor. In § 1.5 we have associated with a Divizor D on a smooth varyety a vector space $\mathcal{L}(D)$.

Theorem 6. The space $\mathcal{L}(D)$ is ?nite–dimensional for any Divizor D on a smooth projektive algebraic curve.

First of all it is easy to reduce the assertion of the theorem to the case $D \geq 0$. For let $D = D_1 - D_2$, $D_1 \geq 0$, $D_2 \geq 0$. Then $\mathcal{L}(D) \subset \mathcal{L}(D_1)$: if $f \in \mathcal{L}(D)$, then $(f) + D_1 - D_2 = D^{\geq 0}$, hence $(f) + D_1 = D^{\geq D_2}$, that is $f \in \mathcal{L}(D_1)$. This gives the required reduction. Now let $D \geq 0$.

$D = \sum_{i=1}^{r} n_i x_i, n_i \geq 0$. At the points x_i we choose local parameters t_i.

The condition $f \in \mathcal{L}(D)$ is equivalent to $v_{x_i}(f) \geq -n_i (i = 1, \ldots, r)$, $v_x(f) \geq 0$ for $x \neq x_i$, that is $f \in t_i^{-n_i}\mathcal{O}_{x_i}$ $(i = 1, \ldots, r)$, $f \in \mathcal{O}_x$ for $x \neq x_i$.

In view of all this we can consider the linear mapping

$$\varphi : \mathcal{L}(D) \to \bigoplus_{i=1}^{r} t_i^{-n_i}\mathcal{O}_{x_i}/\mathcal{O}_{x_i}.$$

that associates with a function $f \in \mathcal{L}(D)$ all its residue classes in the spaces $t_i^{-n_i}\mathcal{O}_{x_i}/\mathcal{O}_{x_i}$. If $\varphi(f) = 0$, then $f \in \mathcal{O}_{x_i}$, $(i = 1, \ldots, r)$, and since $f \in \mathcal{L}(D)$, we have $f \in \mathcal{O}_x$, for $x \neq x_i$. Therefore f is regular at all the points $x \in X$. Since X is a projektive curve, such a function must be a constant. Thus, the kernel of φ is k, hence one-dimensional. To prove that $\mathcal{L}(D)$ is ?nite-dimensional it remains to verify that the space

$\bigoplus_{i=1}^{r} t_i^{-n_i}\mathcal{O}_{x_i}/\mathcal{O}_{x_i}$, is finite-dimensional. Obviously multiplication by $t_i^{n_i}$ determines an isomorphism $t_i^{-n_i}\mathcal{O}_{x_i}/\mathcal{O}_{x_i} \cong \mathcal{O}_{x_i}/t_i^{n_i}\mathcal{O}_{x_i}$ and in the proof of Theorem 2 we have seen that the space $\mathcal{O}_{x_i}/t_i^{n_i}\mathcal{O}_{x_i}$, is of finite dimension n_i. Thus, $\bigoplus_{i=1}^{r} t_i^{-n_i}\mathcal{O}_{x_i}$,

is a direct sum of finite- dimensional spaces.

hence itself ?nite— dimensional

Together with the proof of the theorem we have obtained the estimate

$dim\mathcal{L}(D) \leq deg D + 1$ for $D \geq 0$.

4.2 Commutative algebra approach to Fujita problem

Article is dedicated to Fujita problem [3], one of algebraic geometry problem, which is still open since 1987.

X is smooth projective variety of dimension n. Fujita problem states:

Statement 1. *If X is minimal variety of general type, then linear system $|mK_X|$ has global generation when $m \geq n+2$.*

Statement 2. *If A is ample and invertible sheave on X, then linear system $|mK_X + (n+1)A|$ has global generation and $|mK_X + (n+2)A|$ is very ample on X.* [1]

For surfaces the Fujita conjecture follows from Reider's theorem.

For three-dimensional algebraic varieties Ein and Lazarsfeld in 1993 proved the first part of the Fujita conjecture, i.e. that m>4 implies global generation.

We use commutative algebra.

Commutative algebra approach is using tight closure and allows us to prove theorem in arbitrary characteristic without the use of desingularization or vanishing theorems.

We show that X not be smooth. F-rationality is sufficient.

Also line bundle L not be very ample, it is sufficient that the complete linear system $|L|$ defines a generically finite map to proper subvariety of a projective space of $dim|L|$. [2]

Theorem 1. *X is projective and F-rational and $dim X = d$, complete linear system L defines a generically finite map to proper subvariety of a projective space of dim L then $K_X + dL$ is globally-generated unless $X = P^d$ and L is hyperplane bundle.*

Lemma 1. *The following conditions are equivalent*

1) reflexive sheave O_X is globally generated

2) there exists an integer N such that every element of local cohomology module of H^{d+1}_m of degree less

then N has non-zero multiple of degree $-n$.

Lemma 2. *If a local ring (R, m) of prime characteristic and dimension $d+1$ is F-rational on its punctured spectrum then the tight closure of the zero module in the local cohomology module H^{d+1}_m has finite length.*

Lemma 3. *Let R be a normal N-graded ring over perfect field of prime characteristic p, and let I_1 and I_2 be ideals of R generated by homogeneous elements of degrees strictly less than δ and greater than or equal to δ respectively.Let z be an element of R homogeneous of degree δ. Then if $z \in (I_1 + I_2)^*$, than $z \in I_1^* + I_2$.*

Proof of the Main Theorem 1.

We can assume that section ring S is graded ring of prime characteristic.

X is F-rational follows S is F-rational on its punctured spectrum $SpecS_m$.

By Lemma 1 this means local cohomology module H_m^{d+1} has finite length.

So exists N such that tight closure of the zero lies in submodule, generated by elements of degree N and higher.

So if ν is homogeneous element of H_m^{d+1} of degree$-n<min(N, -d-1)$ then ν is not tight closure of the zero.

We need to show that ν has non-zero multiple of degree -d.

Suppose this is not true. so S_{n-d} kills ν.

Because L globally generated. S admits a system of parameters of degree one $x_0, x_1, \cdots, x_d$.

$$H_m^{d+1}=Cokero:S_{x/x_0} \oplus S_{x/x_1} \cdots \oplus S_{x/x_d} \to S_x$$

where $x=x_0x_1\cdots x_d$

$$\phi(\frac{s_0x_0^t}{x^t}, \cdots, \frac{s_dx_d^t}{x^t}) = \sum_i \frac{(-1)^i s_i x_i^t}{x^t}$$

It is well known fact that an element $[\frac{z}{x^t}]$ is in tight closure of the zero module H_m^{d+1}

if and only if z is in tight closure of the ideal $(x_0^t, x_1^t, \cdots, x_d^t)$ in S.

So we have an element of local cohomology module of type $[\frac{wz}{x^t}]$,

where $w \in (x_0^t, x_1^t, \cdots, x_d^t)^{n-d}$ is equal to zero.

Thus we assume $\nu = \sum \frac{\lambda_{i_0 i_1 \cdots i_d} x_0^{i_0} x_1^{i_1} \cdots x_d^{i_d}}{x^t}$

$$w = x_0^{t-1-i_0} x_1^{t-1-i_1} \cdots x_d^{t-1-i_d} s.$$

We have $degw=n-d$ and $w\nu = [\frac{\lambda s x^{t-1}}{x^t}] = [\frac{\lambda s}{x}] = 0$.

Hence $s \in (x_0^t, x_1^t, \cdots, x_d^t)^*$.

But since s has degree 1 then by Lemma 3 $s \in (x_0^t, x_1^t, \cdots, x_d^t)$.

Using that s is arbitrary element of degree 1 then we can choose s is not in linear system

spanned by x_0. which is possible if $dimH^0(X.L) > d+2$.

If L is very ample then $dimH^0(X,L) > d+2$ except case $L{=}O(4.18)$.

The embedding of X given by complete linear system L would be isomorphism $X \to P^d$.

5. Conclusion

The surfaces of the Fujita hypothesis were confirmed by Igor Reider. In my paper. I generalize the result for X of a non-smooth projective variety, but F-rational.

I am using the methods of commutative algebra and dense closure definitions, if Fujita hypotheses are true for a smooth projective variety, then this is true for the more general case of F-rational ones.

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